Problem 1  Let \( ABC \) be a triangle with \( CA = CB \) and \( \angle ACB = 120^\circ \), and let \( M \) be the midpoint of \( AB \). Let \( P \) be a variable point on the circumcircle of \( ABC \), and let \( Q \) be the point on the segment \( CP \) such that \( QP = 2QC \). It is given that the line through \( P \) and perpendicular to \( AB \) intersects the line \( MQ \) at a unique point \( N \).

Prove that there exists a fixed circle such that \( N \) lies on this circle for all possible positions of \( P \).

(Velina Ivanova, Bulgaria)

Solution  Let \( O \) be the circumcenter of \( ABC \). From the assumption that \( \angle ACB = 120^\circ \) it follows that \( M \) is the midpoint of \( CO \).

Let \( \omega \) denote the circle with center in \( C \) and radius \( CO \). This circle in the image of the circumcircle of \( ABC \) through the translation that sends \( O \) to \( C \). We claim that \( N \) lies on \( \omega \).

Let us consider the triangles \( QNP \) and \( QMC \). The angles in \( Q \) are equal. Since \( NP \) is parallel to \( MC \) (both lines are perpendicular to \( AB \)), it turns out that \( \angle QNP = \angle QMC \), and hence the two triangles are similar. Since \( QP = 2QC \), it follows that 

\[ NP = 2MC = CO, \]

which proves that \( N \) lies on \( \omega \).

Comment  The possible positions of \( N \) are all the points of \( \omega \) with the exception of the two points lying on the line \( CO \). Indeed, \( P \) does not lie on the line \( CO \) because otherwise the point \( N \) is not well-defined, and therefore also \( N \) does not lie on the same line.

Conversely, let \( N \) be any point on \( \omega \) and not lying on the line \( CO \). Let \( P \) be the corresponding point on the circumcircle of \( ABC \), namely such that \( NP \) is parallel and equal to \( CO \). Let \( Q \) be
the intersection of $CP$ and $NM$. As before, the triangles $QNP$ and $QMC$ are similar, and now from the relation $NP = 2MC$ we deduce that $QP = 2QC$. This proves that $N$ can be obtained from $P$ through the construction described in the statement of the problem.

**Alternative solution** Let $M'$ denote the symmetric of $M$ with respect to $O$.

Let us consider the quadrilateral $MM'PN$. The lines $MM'$ and $NP$ are parallel by construction. Also the lines $PM'$ and $NM$ are parallel (homothety from $C$ with coefficient 3). It follows that $MM'PN$ is a parallelogram, and hence $PN = MM' = OC$.

**Computational solution** There are many computation approaches to this problem. For example, we can set Cartesian coordinates so that

$$A = \left( -\frac{\sqrt{3}}{2}, \frac{1}{2} \right), \quad B = \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right), \quad C = (0, 1), \quad M = \left( 0, \frac{1}{2} \right).$$

Setting $P = (a, b)$, we obtain that $Q = (a/3, (2 + b)/3)$. The equation of the line through $P$ and perpendicular to $AB$ is $x = a$. The equation of the line $MQ$ (if $a \neq 0$) is

$$y - \frac{1}{2} = \frac{x}{a} \left( \frac{1}{2} + b \right).$$

The intersection of the two lines is therefore

$$N = (a, 1 + b) = P + (0, 1).$$

This shows that the map $P \to N$ in the translation by the vector $(0, 1)$. This result is independent of the position of $P$ (provided that $a \neq 0$, because otherwise $N$ is not well-defined).

When $P$ lies on the circumcircle of $ABC$, with the exception of the two points with $a = 0$, then necessarily $N$ lies on the translated circle (which is the circle with center in $C$ and radius 1).
Problem 2  Consider the set

\[ A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, \ldots \right\}. \]

(a) Prove that every integer \( x \geq 2 \) can be written as the product of one or more elements of \( A \), which are not necessarily different.

(b) For every integer \( x \geq 2 \), let \( f(x) \) denote the minimum integer such that \( x \) can be written as the product of \( f(x) \) elements of \( A \), which are not necessarily different.

Prove that there exist infinitely many pairs \((x, y)\) of integers with \( x \geq 2, y \geq 2 \), and

\[ f(xy) < f(x) + f(y). \]

(Pairs \((x_1, y_1)\) and \((x_2, y_2)\) are different if \( x_1 \neq x_2 \) or \( y_1 \neq y_2 \)).

(Mihail Baluna, Romania)

Solution  Every integer \( x \geq 2 \) can be written as the telescopic product of \( x - 1 \) elements of \( A \) as

\[ x = \left(1 + \frac{1}{x-1}\right) \cdot \left(1 + \frac{1}{x-2}\right) \cdot \ldots \cdot \left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{1}\right), \]

which is enough to establish part (a). We now consider part (b). Notice that for any positive integer \( k \) we have

\[ f(2^k + 1) \leq k + 1, \]

because \( 2^k + 1 = \left(1 + \frac{1}{2^k}\right) \cdot 2^k \) is a representation of \( 2^k + 1 \) as a product of \( k + 1 \) elements of \( A \).

We claim that all the pairs \((x, y)\) of the form

\[ x = 5, \quad y = \frac{2^{4k+2} + 1}{5} \]

satisfy the required inequality. Notice that \( y \) is an integer for any positive value of \( k \), because \( 2^{4k+2} + 1 = 16^k \cdot 4 + 1 \equiv 5 \equiv 0 \pmod{5} \). Furthermore, \( f(xy) = f(2^{4k+2} + 1) \leq 4k + 3 \) (and \( f(x) = f(2^2 + 1) \leq 3 \)) by the above. We now need some lower bounds on the values of \( f \). Notice that no element of \( A \) exceeds 2, and therefore the product of at most \( k \) elements of \( A \) does not exceed \( 2^k \): it follows that

\[ f(n) \geq \lceil \log_2(n) \rceil, \tag{Q2.1} \]

and in particular that

\[ f(5) = f(2^2 + 1) \geq \lceil \log_2(5) \rceil = 3. \]

We have thus proven \( f(x) = f(5) = 3 \). We want to show \( f(xy) < f(x) + f(y) \), and since we know \( f(xy) \leq 4k + 3 \) and \( f(x) = 3 \) we are reduced to showing \( f(y) > 4k \). Since \( y > 2^{4k-1} \), from (Q2.1) we already know that \( f(y) \geq 4k \), and hence we just need to exclude that \( f(y) = 4k \). Let us assume that we can represent \( y \) in the form \( a_1 \cdot \ldots \cdot a_{4k} \) with every \( a_i \) in \( A \). At least one of the \( a_i \) is not 2 (otherwise the product would be a power of 2, while \( y \) is odd), and hence it is less than or equal to 3/2. It follows that

\[ a_1 \cdot \ldots \cdot a_{4k} \leq 2^{4k-1} \cdot 3 \cdot \frac{3}{2} = 15 \cdot \frac{2^{4k-2}}{5} < \frac{2^{4k+2}}{5} < y, \]

3
which contradicts the fact that \(a_1 \cdots a_{4k}\) is a representation of \(y\).

Note. Using a similar approach one can also prove that all pairs of the form

\[
\left(3, \frac{2^{2k+1} + 1}{3}\right) \quad \text{and} \quad \left(11, \frac{2^{10k+5} + 1}{11}\right)
\]
satisfy the required inequality.

**Second solution** As in the previous solution we obtain the lower bound (Q2.1).

Now we claim that all the pairs of the form

\[x = 2^k + 1, \quad y = 4^k - 2^k + 1\]
satisfy the required inequality when \(k\) is large enough. To begin with, it is easy to see that

\[
2^k + 1 = \frac{2^k + 1}{2^k} \cdot \underbrace{2 \cdots 2}_{k \text{ terms}}
\]
and

\[
2^{3k} + 1 = \frac{2^{3k} + 1}{2^{3k}} \cdot \underbrace{2 \cdots 2}_{3k \text{ terms}}
\]
which shows that \(f(2^k + 1) \leq k + 1\) and \(f(2^{3k} + 1) \leq 3k + 1\). On the other hand, from (Q2.1) we deduce that the previous inequalities are actually equalities, and therefore

\[f(x) = k + 1 \quad \text{and} \quad f(xy) = 3k + 1.
\]

Therefore, it remain to show that \(f(y) > 2k\). Since \(y > 2^{2k-1}\) (for \(k \geq 1\)), from (Q2.1) we already know that \(f(y) \geq 2k\), and hence we just need to exclude that \(f(y) = 2k\). Let us assume that we can represent \(y\) in the form \(a_1 \cdots a_{2k}\). At least one of the factors is not 2, and hence it is less than or equal to \(3/2\). Thus when \(k\) is large enough it follows that

\[a_1 \cdots a_{2k} \leq 2^{2k-1} \cdot \frac{3}{2} = \frac{3}{4} \cdot 2^{2k} < 2^{2k} - 2^k < y,
\]
which contradicts the fact that \(a_1 \cdots a_{2k}\) is a representation of \(y\).

**Third solution** Let’s start by showing that \((x, y) = (7, 7)\) satisfies \(f(xy) < f(x) + f(y)\). We have \(f(7) \geq 4\) since 7 cannot be written as the product of 3 or fewer elements of \(A\): indeed \(2^3 > 7\), and any other product of at most three elements of \(A\) does not exceed \(2^2 \cdot \frac{3}{2} = 6 < 7\). On the other hand, \(f(49) \leq 7\) since \(49 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot \frac{49}{35}\).

Suppose by contradiction that there exist only finitely many pairs \((x, y)\) that satisfy \(f(xy) < f(x) + f(y)\). This implies that there exists \(M\) large enough so that whenever \(a > M\) or \(b > M\) holds we have \(f(ab) = f(a) + f(b)\) (indeed, it is clear that the reverse inequality \(f(ab) \leq f(a) + f(b)\) is always satisfied).

Now take any pair \((x, y)\) that satisfies \(f(xy) < f(x) + f(y)\) and let \(n > M\) be any integer. We obtain

\[f(n) + f(xy) = f(nx) = f(nx) + f(y) = f(n) + f(x) + f(y),\]
which contradicts \(f(xy) < f(x) + f(y)\).
Problem 3  The $n$ contestants of an EGMO are named $C_1, \ldots , C_n$. After the competition they queue in front of the restaurant according to the following rules.

- The Jury chooses the initial order of the contestants in the queue.
- Every minute, the Jury chooses an integer $i$ with $1 \leq i \leq n$.
  - If contestant $C_i$ has at least $i$ other contestants in front of her, she pays one euro to the Jury and moves forward in the queue by exactly $i$ positions.
  - If contestant $C_i$ has fewer than $i$ other contestants in front of her, the restaurant opens and the process ends.

(a) Prove that the process cannot continue indefinitely, regardless of the Jury’s choices.

(b) Determine for every $n$ the maximum number of euros that the Jury can collect by cunningly choosing the initial order and the sequence of moves.

(Hungary)

Solution  The maximal number of euros is $2^n - n - 1$.

To begin with, we show that it is possible for the Jury to collect this number of euros. We argue by induction. Let us assume that the Jury can collect $M_n$ euros in a configuration with $n$ contestants. Then we show that the Jury can collect at least $2M_n + n$ moves in a configuration with $n + 1$ contestants. Indeed, let us begin with all the contestants lined up in reverse order. In the first $M_n$ moves the Jury keeps $C_{n+1}$ in first position and reverses the order of the remaining contestants, then in the next $n$ moves all contestants $C_1, \ldots , C_n$ (in this order) jump over $C_{n+1}$ and end up in the first $n$ positions of the line in reverse order, and finally in the last $M_n$ moves the Jury rearranges the first $n$ positions.

Since $M_1 = 0$ and $M_{n+1} \geq 2M_n + n$, an easy induction shows that $M_n \geq 2^n - n - 1$.

Let us show now that at most $2^n - n - 1$ moves are possible. To this end, let us identify a line of contestants with a permutation $\sigma$ of $\{1, \ldots , n\}$. To each permutation we associate the set of reverse pairs

$$R(\sigma) := \{(i, j) : 1 \leq i < j \leq n \text{ and } \sigma(i) \gt \sigma(j)\},$$

and the nonnegative integer

$$W(\sigma) := \sum_{(i, j) \in R(\sigma)} 2^i,$$

which we call the total weight of the permutation. We claim that the total weight decreases after any move of the contestants. Indeed, let us assume that $C_i$ moves forward in the queue, let $\sigma$ be the permutation before the move, and let $\sigma'$ denote the permutation after the move. Since $C_i$ jumps over exactly $i$ contestants, necessarily she jumps over at least one contestant $C_j$ with index
\( j > i \). This means that the pair \((i, j)\) is reverse with respect to \(\sigma\) but not with respect to \(\sigma'\), and this yields a reduction of \(2^i\) in the total weight. On the other hand, the move by \(C_i\) can create new reverse pairs of the form \((k, i)\) with \(k < i\), but their total contribution is at most

\[
2^0 + 2^1 + \ldots + 2^{i-1} = 2^i - 1.
\]

In conclusion, when passing from \(\sigma\) to \(\sigma'\), at least one term \(2^i\) disappears from the computation of the total weight, and the sum of all the new terms that might have been created is at most \(2^i - 1\). This shows that \(W(\sigma') \leq W(\sigma) - 1\).

We conclude by observing that the maximum possible value of \(W(\sigma)\) is realized when all pairs are reverse, in which case

\[
W(\sigma) = \sum_{i=1}^{n} (i - 1)2^i = 2^n - n - 1.
\]

This proves that the number of moves is less than or equal to \(2^n - n - 1\), and in particular it is finite.

**Alternative solution** As in the previous solution, the fundamental observation is again that, when a contestant \(C_i\) moves forward, necessarily she has to jump over at least one contestant \(C_j\) with \(j > i\).

Let us now show that the process ends after a finite number of moves. Let us assume that this is not the case. Then at least one contestant moves infinitely many times. Let \(i_0\) be the largest index such that \(C_{i_0}\) moves infinitely many times. Then necessarily \(C_{i_0}\) jumps infinitely many times over some fixed \(C_{j_0}\) with \(j_0 > i_0\). On the other hand, we know that \(C_{j_0}\) makes only a finite number of moves, and therefore she can precede \(C_{i_0}\) in the line only a finite number of times, which is absurd.

In order to estimate from above the maximal number of moves, we show that the contestant \(C_i\) can make at most \(2^n - i - 1\) moves. Indeed, let us argue by “backward extended induction”. To begin with, we observe that the estimate is trivially true for \(C_n\) because she has no legal move.

Let us assume now that the estimate has been proved for \(C_i, C_{i+1}, \ldots, C_n\), and let us prove it for \(C_{i-1}\). When \(C_{i-1}\) moves, at least one contestant \(C_j\) with \(j > i - 1\) must precede her in the line. The initial configuration can provide at most \(n - i\) contestants with larger index in front of \(C_{i-1}\), which means at most \(n - i\) moves for \(C_{i-1}\). All other moves are possible only if some contestant in the range \(C_i, C_{i+1}, \ldots, C_n\) jumps over \(C_{i-1}\) during her moves. As a consequence, the total number of moves of \(C_{i-1}\) is at most

\[
n - i + \sum_{k=i}^{n} (2^{n-k} - 1) = 2^{n-i+1} - 1.
\]

Summing over all indices we obtain that

\[
\sum_{i=1}^{n} (2^{n-i} - 1) = 2^n - n - 1,
\]

which gives an estimate for the total number of moves.

The same example of the first solution shows that this upper bound can actually be achieved.

**Comment** In every move of the example, the moving contestant jumps over exactly one contestant with larger index (and as a consequence over all contestants with smaller index).
**Problem 4**  A *domino* is a $1 \times 2$ or $2 \times 1$ tile.

Let $n \geq 3$ be an integer. Dominoes are placed on an $n \times n$ board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap.

The *value* of a row or column is the number of dominoes that cover at least one cell of this row or column. The configuration is called *balanced* if there exists some $k \geq 1$ such that each row and each column has a value of $k$.

Prove that a balanced configuration exists for every $n \geq 3$, and find the minimum number of dominoes needed in such a configuration.

(Merlijn Staps, The Netherlands)

**Solution**  The minimal number of dominoes required in a balanced configuration is $2n/3$ if $n$ is a multiple of 3, and $2n$ otherwise.

In order to show that this number is necessary, we count in two different ways the number of elements of the set $S$ of all pairs $(\ell, d)$, where $\ell$ is a row or a column of the board, and $d$ is a domino that covers at least one cell of that row or column. On the one hand, since each row or column intersects the same number $k$ of dominoes, the set $S$ has $2nk$ elements. On the other hand, since each domino intersects 3 rows/columns, the set $S$ has $3D$ elements, where $D$ is the total number of dominoes on the board. This leads to the equality

$$2nk = 3D.$$

If $n$ is a multiple of 3, from the trivial inequality $k \geq 1$ we obtain that $D \geq 2n/3$. If $n$ is not a multiple of 3, then $k$ is a multiple of 3, which means that $k \geq 3$ and hence $D \geq 2n$.

Now we need to exhibit a balanced configuration with this number of dominoes. The following diagram shows a balanced configuration with $n = 3$ and $k = 1$.

![Balanced Configuration](image)

If $n$ is any multiple of 3, we can obtain a balanced configuration with $k = 1$ by using $n/3$ of these $3 \times 3$ blocks along the principal diagonal of the board.

The following diagrams show balanced configurations with $k = 3$ and $n \in \{4, 5, 6, 7\}$.

![Balanced Configurations](images)

Any $n \geq 8$ can be written in the form $4A + r$ where $A$ is a positive integer and $r \in \{4, 5, 6, 7\}$. Therefore, we can obtain a balanced configuration with $n \geq 8$ and $k = 3$ by using one block with size $r \times r$, and $A$ blocks with size $4 \times 4$ along the principal diagonal of the board. In particular, this construction covers all the cases where $n$ is not a multiple of 3.
Problem 5 Let Γ be the circumcircle of triangle \( ABC \). A circle \( \Omega \) is tangent to the line segment \( AB \) and is tangent to Γ at a point lying on the same side of the line \( AB \) as \( C \). The angle bisector of \( \angle BCA \) intersects \( \Omega \) at two different points \( P \) and \( Q \).

Prove that \( \angle ABP = \angle QBC \).

(Dominika Regieć, Poland)

Solution 1 Let \( M \) be the midpoint of the arc \( AB \) that does not contain \( C \), let \( V \) be the intersection of \( \Omega \) and Γ, and let \( U \) be the intersection of \( \Omega \) and \( AB \).

The proof can be divided in two steps:

1. Proving that \( MP \cdot MQ = MB^2 \).

   It is well-known that \( V, U \) and \( M \) are collinear (indeed the homothety with center in \( V \) that sends \( \Omega \) to Γ sends \( U \) to the point of Γ where the tangent to Γ is parallel to \( AB \), and this point is \( M \)), and

   \[
   MV \cdot MU = MA^2 = MB^2.
   \]

   This follows from the similitude between the triangles \( \Delta MAV \) and \( \Delta MUA \). Alternatively, it is a consequence of the following well-known lemma: *Given a circle \( \Gamma \) with a chord \( AB \), let \( M \) be the middle point of one of the two arcs \( AB \). Take a line through \( M \) which intersects \( \Gamma \) again at \( X \) and \( AB \) at \( Y \). Then \( MX \cdot MY \) is independent of the choice of the line.*

   Computing the power of \( M \) with respect to \( \Omega \) we obtain that

   \[
   MP \cdot MQ = MU \cdot MV = MB^2.
   \]

2. Conclude the proof given that \( MP \cdot MQ = MB^2 \).

   The relation \( MP \cdot MQ = MB^2 \) in turn implies that triangle \( \Delta MBP \) is similar to triangle \( \Delta MQB \), and in particular \( \angle MBP = \angle MQB \). Keeping into account that \( \angle MCB = \angle MBA \), we finally conclude that

   \[
   \angle QBC = \angle MQB - \angle MCB = \angle MBP - \angle MBA = \angle PBA,
   \]

   as required.
Solution 2  The second solution is in fact a different proof of the first part of Solution 1.

Let us consider the inversion with respect to circle with center $M$ and radius $MA = MB$. This inversion switches $AB$ and $\Gamma$, and fixes the line passing through $M, U, V$. As a consequence, it keeps $\Omega$ fixed, and therefore it switches $P$ and $Q$. This is because they are the intersections between the fixed line $MC$ and $\Omega$, and the only fixed point on the segment $MC$ is its intersection with the inversion circle (thus $P$ and $Q$ are switched). This implies that $MP \cdot MQ = MB^2$.

Solution 3  This solution is instead a different proof of the second step of Solution 1.

Let $I$ and $J$ be the incenter and the $C$-excenter of $\triangle ABC$ respectively. It is well-known that $MA = MI = MJ$, therefore the relation $MP \cdot MQ = MA^2$ implies that $(P, Q, I, J) = -1$.

Now observe that $\angle IBJ = 90^\circ$, thus $BI$ is the angle bisector of $\angle PBQ$ as it is well-known from the theory of harmonic pencils, and this leads easily to the conclusion.

Solution 4  Let $D$ denote the intersection of $AB$ and $CM$. Let us consider an inversion with respect to $B$, and let us use primes to denote corresponding points in the transformed diagram, with the gentlemen agreement that $B' = B$.

Since inversion preserves angles, it turns out that

$$\angle A'B'M' = \angle A'M'B' = \angle ACB,$$

and in particular triangle $A'B'M'$ is isosceles with basis $B'M'$.

The image of $CM$ is the circumcircle of $B'C'M'$, which we denote by $\omega'$. It follows that the centers of both $\omega'$ and the image $\Omega'$ of $\Omega$ lie on the perpendicular bisector of $B'M'$. Therefore, the whole transformed diagram is symmetric with respect to the perpendicular bisector of $B'M'$, and in particular the arcs $D'P'$ and $Q'C'$ of $\omega'$ are equal.

This is enough to conclude that $\angle D'B'P' = \angle Q'B'C'$, which implies the conclusion.
Problem 6

(a) Prove that for every real number $t$ such that $0 < t < \frac{1}{2}$ there exists a positive integer $n$ with the following property: for every set $S$ of $n$ positive integers there exist two different elements $x$ and $y$ of $S$, and a non-negative integer $m$ (i.e. $m \geq 0$), such that

$$|x - my| \leq ty.$$ 

(b) Determine whether for every real number $t$ such that $0 < t < \frac{1}{2}$ there exists an infinite set $S$ of positive integers such that

$$|x - my| > ty$$

for every pair of different elements $x$ and $y$ of $S$ and every positive integer $m$ (i.e. $m > 0$).

(Merlijn Staps, The Netherlands)

Solution

Part (a) Let $n$ be any positive integer such that

$$(1 + t)^{n-1} \geq \frac{1}{t} \quad \text{(Q6.1)}$$

(this inequality is actually true for every large enough $n$ due to Bernoulli’s inequality).

Let $S$ be any set of $n$ distinct positive integers, which we denote by

$$s_1 < s_2 < \ldots < s_n.$$ 

We distinguish two cases.

- If $s_{i+1} \leq (1 + t)s_i$ for some $i \in \{1, \ldots, n-1\}$, then

$$|s_{i+1} - s_i| = s_{i+1} - s_i \leq ts_i,$$

and therefore the required inequality is satisfied with $x = s_{i+1}$, $y = s_i$, and $m = 1$.

- If $s_{i+1} > (1 + t)s_i$ for every $i \in \{1, \ldots, n-1\}$, then by induction we obtain that

$$s_n > (1 + t)^{n-1}s_1.$$ 

As a consequence, from (Q6.1) it follows that

$$|s_1| = s_1 < \frac{1}{(1 + t)^{n-1}} \cdot s_n \leq ts_n,$$

and therefore the required inequality is satisfied with $x = s_1$, $y = s_n$, and $m = 0$. 

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Part (b) (Explicit formula) We claim that an infinite set with the required property exists. To this end, we rewrite the required condition in the form

$$\left| \frac{x}{y} - m \right| > t.$$  

This is equivalent to saying that the distance between the ratio $x/y$ and the set of positive integers is greater than $t$.

Now we construct an increasing sequence $s_n$ of odd coprime positive integers satisfying

$$\frac{1}{2} - \frac{1}{2s_n} > t \quad \forall n \geq 1,$$  

(Q6.2)

and such that for every $j > i$ it turns out that

$$\frac{s_i}{s_j} < \frac{1}{2} \quad \text{and} \quad t < \left\{ \frac{s_j}{s_i} \right\} < \frac{1}{2},$$  

(Q6.3)

where $\{\alpha\}$ denotes the fractional part of $\alpha$. This is enough to show that the set $S := \{s_n : n \geq 1\}$ has the required property.

To this end, we consider the sequence defined recursively by

$$s_{n+1} = \frac{(s_1 \cdots s_n)^2 + 1}{2},$$

with $s_1$ large enough. An easy induction shows that this is an increasing sequence of odd positive integers. For every $i \in \{1, \ldots, n\}$ it turns out that

$$\frac{s_i}{s_{n+1}} \leq \frac{2}{s_i} \leq \frac{2}{s_1} < \frac{1}{2}$$

because $s_1$ is large enough, which proves the first relation in (Q6.3). Moreover, it turns out that

$$\frac{s_{n+1}}{s_i} = \frac{(s_1 \cdots s_n)^2}{2s_i} + \frac{1}{2s_i}.$$  

The first term is a positive integer plus $1/2$, from which it follows that the distance of $s_{n+1}/s_i$ from the positive integers is greater than or equal to

$$\frac{1}{2} - \frac{1}{2s_i} \geq \frac{1}{2} - \frac{1}{2s_1},$$

which is greater than $t$ if $s_1$ is large enough. This proves the second relation in (Q6.3).

Part (b) (Arithmetic approach) We produce an increasing sequence $s_n$ of odd and coprime positive integers that satisfies (Q6.3) every $j > i$. As in the previous solution, this is enough to conclude.

We argue by induction. To begin with, we choose $s_1$ to be any odd integer satisfying the inequality in (Q6.2). Let us assume now that $s_1, \ldots, s_n$ have already been chosen, and let us choose $s_{n+1}$ in such a way that

$$s_{n+1} \equiv \frac{s_i - 1}{2} \pmod{s_i} \quad \forall i \in \{1, \ldots, n\}.$$
We can solve this system because the previously chosen integers are odd and coprime. Moreover, any solution of this system is coprime with $s_1, \ldots, s_n$. Indeed, for every $1 \leq i \leq n$ it turns out that

$$s_{n+1} = \frac{s_i - 1}{2} + k_i s_i$$

for some positive integer $k_i$. Therefore, any prime $p$ that divides both $s_{n+1}$ and $s_i$ divides also $(2k_i + 1)s_i - 2s_{n+1} = 1$, which is absurd. Finally, we observe that we can assume that $s_{n+1}$ is odd and large enough. In this way we can guarantee that

$$\frac{s_i}{s_{n+1}} < \frac{1}{2} \quad \forall i \in \{1, \ldots, n\},$$

which is the first requirement in (Q6.3), and

$$k_i + t < k_i + \frac{1}{2} - \frac{1}{2s_i} = \frac{s_{n+1}}{s_i} < k_i + \frac{1}{2} \quad \forall i \in \{1, \ldots, n\},$$

which implies the second requirement in (Q6.3).

**Part (b) (Algebraic approach)** Again we produce an increasing sequence $s_n$ of positive integers that satisfies (Q6.3) every $j > i$.

To this end, for every positive integer $x$, we define its security region $S(x) := \bigcup_{n \geq 1} ((n + t)x, (n + \frac{1}{2})x)$.

The security region $S(x)$ is a periodic countable union of intervals of length $(\frac{1}{2} - t)x$, whose left-hand or right-hand endpoints form an arithmetic sequence. It has the property that

$$t < \left\{ \frac{y}{x} \right\} < \frac{1}{2} \quad \forall y \in S(x).$$

Now we prove by induction that we can choose a sequence $s_n$ of positive integers satisfying (Q6.3) and in addition the fact that every interval of the security region $S(s_n)$ contains at least one interval of $S(s_{n-1})$.

To begin with, we choose $s_1$ large enough so that the length of the intervals of $S(s_1)$ is larger than 1. This guarantees that any interval of $S(s_1)$ contains at least a positive integer. Now let us choose a positive integer $s_2 \in S(s_1)$ that is large enough. This guarantees that $s_1/s_2$ is small enough, that the fractional part of $s_2/s_1$ is in $(t, 1/2)$, and that every interval of the security region $S(s_2)$ contains at least one interval of $S(s_1)$, and hence at least one positive integer.

Let us now assume that $s_1, \ldots, s_n$ have been already chosen with the required properties. We know that every interval of $S(s_n)$ contains at least one interval of $S(s_{n-1})$, which in turn contains an interval in $S(s_{n-2})$, and so on up to $S(s_1)$. As a consequence, we can choose a large enough positive integer $s_{n+1}$ that lies in $S(s_k)$ for every $k \in \{1, \ldots, n\}$. Since $s_{n+1}$ is large enough, we are sure that

$$\frac{s_k}{s_{n+1}} < t \quad \forall k \in \{1, \ldots, n\}.$$  

Moreover, we are sure also that all the intervals of $S(s_{n+1})$ are large enough, and therefore they contain at least one interval of $S(s_n)$, which in turn contains at least one interval of $S(s_{n-1})$, and so on. Finally, the condition

$$t < \left\{ \frac{s_{n-1}}{s_n} \right\} < \frac{1}{2}$$

is guaranteed by the fact that $s_{n+1}$ was chosen in an interval that is contained in $S(s_k)$ for every $k \in \{1, \ldots, n\}$. This completes the induction.