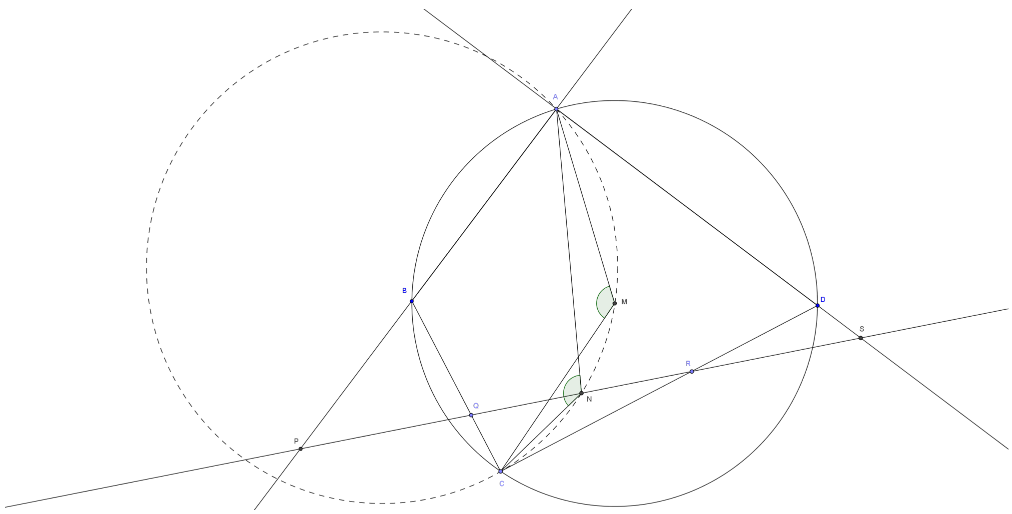
Problem 1

Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BCD = 90^\circ$ and $\angle ABC > \angle CDA$. Let Q and R be points on the segments BC and CD, respectively, such that the line QR intersects lines AB and AD at points P and S, respectively. It is given that $PQ = RS$. Let the midpoint of BD be M and the midpoint of QR be N. Prove that M, N, A and C lie on a circle.

Mark Mordechai Etkind, Israel

Solution 1

Note that N is also the midpoint of PS. From right-angled triangles PAS and CQR we obtain $\angle ANP = 2\angle ASP$, $\angle CNQ = 2\angle CRQ$, hence $\angle ANC = \angle ANP + \angle CNQ = 2(\angle ASP + \angle CRQ) = 2(\angle RSD + \angle DRS) = 2\angle ADC$.
Similarly, using right-angled triangles BAD and BCD, we obtain $\angle AMC = 2\angle ADC$.
Thus $\angle AMC = \angle ANC$, and the required statement follows.



Solution 2

In this proof we show that we have $\angle NCM = \angle NAM$ instead. From right-angled triangles BCD and QCR we get $\angle DRS = \angle CRQ = \angle RCN$ and $\angle BDC = \angle DCM$. Hence $\angle NCM = \angle DCM - \angle RCN$. From right-angled triangle APS we get $\angle PSA = \angle SAN$. From right-angled triangle BAD we have $\angle MAD = \angle BDA$. Moreover, $\angle BDA = \angle DRS + \angle RSD - \angle RDB$.
Therefore $\angle NAM = \angle NAS - \angle MAD = \angle CDB - \angle DRS = \angle NCM$, and the required statement follows.

Solution 3

As N is also the midpoint of PS, we can shrink triangle APS to a triangle A_0QR (where P is sent to Q and S is sent to R). Then A_0, Q, R and C lie on a circle with center N. According to the shrinking the line A_0R is parallel to the line AD. Therefore $\angle CNA = \angle CNA_0 = 2\angle CRA_0 = 2\angle CDA = \angle CMA$. The required statement follows.

Problem 2

Find the smallest positive integer k for which there exist a colouring of the positive integers $\mathbb{Z}_{>0}$ with k colours and a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with the following two properties:

(i) For all positive integers m, n of the same colour, $f(m+n) = f(m) + f(n)$.

(ii) There are positive integers m, n such that $f(m+n) \neq f(m) + f(n)$.

In a colouring of $\mathbb{Z}_{>0}$ with k colours, every integer is coloured in exactly one of the k colours. In both (i) and (ii) the positive integers m, n are not necessarily different.

Merlijn Staps, the Netherlands

Solution 1:

The answer is $k = 3$.
First we show that there is such a function and coloring for $k = 3$. Consider $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ given by $f(n) = n$ for all $n \equiv 1$ or 2 modulo 3, and $f(n) = 2n$ for $n \equiv 0$ modulo 3. Moreover, give a positive integer n the i-th color if $n \equiv i$ (3).
By construction we have $f(1+2) = 6 \neq 3 = f(1) + f(2)$ and hence f has property (ii).
Now let n, m be positive integers with the same color i. If $i = 0$, then $n + m$ has color 0, so $f(n+m) = 2(n+m) = 2n + 2m = f(n) + f(m)$. If $i = 1$, then $n + m$ has color 2, so $f(n+m) = n + m = f(n) + f(m)$. Finally, if $i = 2$, then $n + m$ has color 1, so $f(n+m) = n + m = f(n) + f(m)$. Therefore f also satisfies condition (i).

Next we show that there is no such function and coloring for $k = 2$.
Consider any coloring of $\mathbb{Z}_{>0}$ with 2 colors and any function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ satisfying conditions (i) and (ii). Then there exist positive integers m and n such that $f(m+n) \neq f(m) + f(n)$. Choose m and n such that their sum is minimal among all such m, n and define $a = m + n$. Then in particular for every $b < a$ we have $f(b) = bf(1)$ and $f(a) \neq af(1)$.
If a is even, then condition (i) for $m = n = \frac{a}{2}$ implies $f(a) = f(\frac{a}{2}) + f(\frac{a}{2}) = f(1)a$, a contradiction. Hence a is odd. We will prove two lemmas.

Lemma 1. *Any odd integer $b < a$ has a different color than a.*

Proof. Suppose that $b < a$ is an odd integer, and that a and b have the same color. Then on the one hand, $f(a+b) = f(a) + bf(1)$. On the other hand, we also have $f(a+b) = f(\frac{a+b}{2}) + f(\frac{a+b}{2}) = (a+b)f(1)$, as $\frac{a+b}{2}$ is a positive integer smaller than a. Hence $f(a) = f(a+b) - bf(1) = (a+b)f(1) - bf(1) = af(1)$, which is again a contradiction. Therefore all odd integers smaller than a have a color different from that of a. $\square$

Lemma 2. *Any even integer $b < a$ has the same color as a*

Proof. Suppose $b < a$ is an even integer, and that a and b have different colors. Then $a - b$ is an odd integer smaller than a, so it has the same color as b. Thus $f(a) = f(a-b) + f(b) = (a-b)f(1) + bf(1) = af(1)$, a contradiction. Hence all even integers smaller than a have the same color as a. $\square$

Suppose now $a + 1$ has the same color as a. As $a > 1$, we have $\frac{a+1}{2} < a$ and therefore $f(a+1) = 2f(\frac{a+1}{2}) = (a+1)f(1)$. As $a - 1$ is an even integer smaller than a, we have by Lemma 2 that $a - 1$ also has the same color as a. Hence $2f(a) = f(2a) = f(a+1) + f(a-1) = (a+1)f(1) + (a-1)f(1) = 2af(1)$, which implies that $f(a) = af(1)$, a contradiction. So a and $a + 1$ have different colors.
Since $a - 2$ is an odd integer smaller than a, by Lemma 1 it has a color different from that of a, so $a - 2$ and $a + 1$ have the same color. Also, we have seen by Lemma 2 that $a - 1$ and a have the same color. So $f(a) + f(a-1) = f(2a-1) = f(a+1) + f(a-2) = (a+1)f(1) + (a-2)f(1) = (2a-1)f(1)$, from which it follows that $f(a) = (2a-1)f(1) - f(a-1) = (2a-1)f(1) - (a-1)f(1) = af(1)$, which contradicts our choice of a and finishes the proof.

Solution 2:

We prove that $k \leq 3$ just as in first solution.
Next we show that there is no such function and coloring for $k = 2$.
Consider any coloring of $\mathbb{Z}_{>0}$ with 2 colors and any function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ satisfying conditions (i) and (ii).
We first notice with $m = n$ that $f(2n) = 2f(n)$.

Lemma 3. *For every $n \in \mathbb{Z}_{>0}$, $f(3n) = 3f(n)$ holds.*

Proof. Define $c = f(n), d = f(3n)$. Then we have the relations

$$f(2n) = 2c, \qquad f(4n) = 4c, \qquad f(6n) = 2d.$$

- If n and $2n$ have the same color, then $f(3n) = f(n) + f(2n) = 3c = 3f(n)$.
- If n and $3n$ have the same color, then $4c = f(4n) = f(n) + f(3n) = c + f(3n)$, so $f(3n) = 3f(n)$.
- If $2n$ and $4n$ have the same color, then $2d = f(6n) = f(2n) + f(4n) = 2c + 4c = 6c$, so $f(3n) = d = 3c$.
- Otherwise n and $4n$ have the same color, and $2n$ and $3n$ both have the opposite color to n. Therefore we compute $5c = f(n) + f(4n) = f(5n) = f(2n) + f(3n) = 2c + f(3n)$ so $f(3n) = 3f(n)$.

Consequently, for $k = 2$ we necessarily have $f(3n) = 3f(n)$. $\square$

Now let a be the smallest integer such that $f(a) \neq af(1)$. In particular a is odd and $a > 3$. Consider the three integers a, $\frac{a-3}{2}$, $\frac{a+3}{2}$. By pigeonhole principle two of them have the same color.

- If $\frac{a-3}{2}$ and $\frac{a+3}{2}$ have the same color, then $f(a) = \frac{a-3}{2}f(1) + \frac{a+3}{2}f(1) = af(1)$.
- If a and $\frac{a-3}{2}$ have the same color, then $3\frac{a-1}{2}f(1) = 3f(\frac{a-1}{2}) = f(\frac{3a-3}{2}) = f(a) + f(\frac{a-3}{2}) = f(a) + \frac{a-3}{2}f(1)$, so $f(a) = af(1)$.
- If a and $\frac{a+3}{2}$ have the same color, then $3\frac{a+1}{2}f(1) = 3f(\frac{a+1}{2}) = f(\frac{3a+3}{2}) = f(a) + f(\frac{a+3}{2}) = f(a) + \frac{a+3}{2}f(1)$, so $f(a) = af(1)$.

In the three cases we find a contradiction with $f(a) \neq af(1)$, so it finishes the proof.

Solution 3:

As before we prove that $k \leq 3$ and for any such function and colouring we have $f(2n) = 2f(n)$.
Now we show that there is no such function and coloring for $k = 2$.
Consider any coloring of $\mathbb{Z}_{>0}$ with 2 colors and any function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ satisfying conditions (i) and (ii).
Say the two colors are white (W) and black (B). Pick m, n any two integers such that $f(m+n) = f(m)+f(n)$.
Without loss of generality we may assume that $m+n, m$ are black and n is white.

Lemma 4. *For all $l \in \mathbb{Z}_{>0}$ and every x whose color is black, we have $x + lm$ is black and $f(x + lm) = f(x) + lf(m)$.*

Proof. We proceed by induction. It is clearly true for $l = 0$. If $x + lm$ is black and satisfies $f(x + lm) = f(x) + lf(m)$, then $f(x + (l+1)m) = f(x + lm) + f(m) = f(x) + (l+1)f(m)$ and $f(x + (l+1)m + n) = f(x + lm) + f(m+n) = f(x) + lf(m) + f(m+n) \neq f(x) + (l+1)f(m) + f(n) = f(x + (l+1)m) + f(n)$, so $x + (l+1)m$ is not the same color of n, therefore $x + (l+1)m$ is black. Thjs completes the induction. $\square$

In particular we then must have that $2^l n$ is white for every l, because otherwise since $2^l m$ is black we would have $2^l f(m+n) = f(2^l m + 2^l n) = f(2^l m) + f(2^l n) = 2^l(f(m) + f(n))$, and consequently $f(m+n) = f(m) + f(n)$.

Lemma 5. *For every $l \geq 1$, $2^l m + 2^{l-1} n$ is black.*

Proof. On the one hand we have $2^l f(m+n) = f(2^l m + 2^l n) = f(2^{l-1}(2m+n) + 2^{l-1}n)$. On the other hand we have

$$2^l f(m+n) = 2^{l-1} \cdot 2f(m+n) \neq 2^{l-1}(f(m+n)+f(m)+f(n)) = 2^{l-1}(f(2m+n)+f(n)) = f(2^l m+2^{l-1}n))+f(2^{l-1}n).$$

Therefore $2^l m + 2^{l-1}n$ and $2^{l-1}n$ have different color, which means $2^l m + 2^{l-1}n$ is black. $\square$

Combining the two lemmas give $jm + 2^{l-1}n$ is black for all $j \geq 2^l$ and every $l \geq 1$.
Now write $m = 2^{l-1}m'$ with m' odd. Let t be a number such that $\frac{2^t-1}{m'}$ is an integer and $j = \frac{2^t-1}{m'}n \geq 2^l$, i.e. t is some multiple of $\phi(m')$. Then we must have that $jm + 2^{l-1}n$ is black, but by definition $jm + 2^{l-1}n = (2^t-1)2^{l-1}n + 2^{l-1}n = 2^{t+l-1}n$ is white. This is a contradiction, so $k = 2$ is impossible.

Problem 3

There are 2017 lines in a plane such that no 3 of them go through the same point. Turbo the snail can slide along the lines in the following fashion: she initially moves on one of the lines and continues moving on a given line until she reaches an intersection of 2 lines. At the intersection, she follows her journey on the other line turning left or right, alternating the direction she chooses at each intersection point she passes. Can it happen that she slides through a line segment for a second time in her journey but in the opposite direction as she did for the first time?

Márk Di Giovanni, Hungary

1. Solution

We show that this is not possible.

The lines divide the plane into disjoint regions. We claim that there exists an *alternating* 2-coloring of these regions, that is each region can be colored in black or white, such that if two regions share a line segment, they have a different color. We show this inductively.

If there are no lines, this is obvious. Consider now an arrangement of n lines in the plane and an alternating 2-coloring of the regions. If we add a line g, we can simply switch the colors of all regions in one of the half planes divided by g from white to black and vice versa. Any line segment not in g will still be between two regions of different color. Any line segment in g cuts a region determined by the n lines in two, and since we switched colors on one side of g this segment will also lie between two regions of different color.

Now without loss of generality we may assume, that Turbo starts on a line segment with a white region on her left and a black one on her right. At any intersection, if she turns right, she will keep the black tile to her right. If she turns left, she will keep the white tile to her left. Thus wherever she goes, she will always have a white tile on her left and a black tile on her right. As a consequence, she can cross every line segment only in the direction where she has a white tile to her left, and never the opposite direction where she would have a black tile to the left.

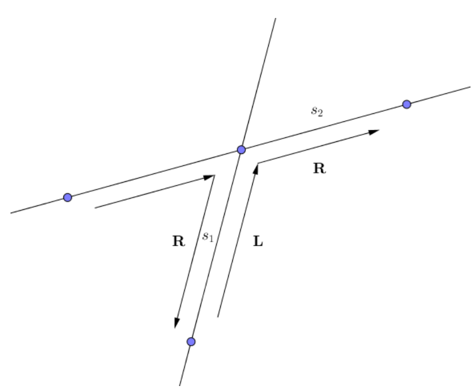
2. Solution

Suppose the assumption is true.

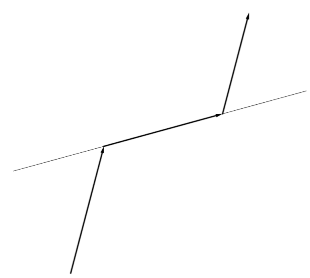
Let's label each segment in the snail's path with **L** or **R** depending on the direction that Turbo chose at the start point of this segment (one segment can have several labels if it has been visited several times).

Consider the first segment that has been visited twice in different directions, name this segment s_1. Assume without loss of generality that it is labeled with **L**. Then next segment must be labeled with **R**, name this one s_2.

Let's look at the label which s_1 can get on the second visit. If it gets **L** then the previous segment in the path must be s_2. But in this case s_1 is not the first segment that has been visited twice in different directions because s_2 has been visited earlier. So the second label of s_1 must be **R**, and Turbo must have come from the opposite side of s_2.



Since Turbo alters directions at each point, labels in her path also alter. And because two labels of s_1 are different, the number of visited segments between these two visits must be even.
Now let's make the following observation: each segment in the path corresponds to exactly one line, and its previous and next segments are on opposite sides of this line.



Again consider the path between two visits of s_1.
Each line intersecting this path must be crossed an even number of times because Turbo has to return to the initial side of each line. Therefore, an even number of segments of Turbo's path are contained on each of these lines. But the line containing s_1 must be crossed an odd number times. Since each crossing corresponds to exactly one segment in the path, the number of segments must be odd.
Here we get the contradiction. Therefore, the assumption is false.

3. Solution

Suppose that the snail always slides slightly to the right of the line segments on her path. When turning to the right, she does not cross any line, whereas when turning to the left, she crosses exactly two lines. This means that at any time of her journey, she has crossed an even number of lines.
Assuming that at some point she slides along a segment for the second time, but in the opposite direction, we argue that she needs to cross an odd number of lines. Let ℓ be the line on which the revisit happens. In order to get to the other side of ℓ, the snail has to cross ℓ an odd number of times. To visit the same segment of ℓ, she must cross every other line an even number of times.

4. Solution

Let us color in red all intersection points of the given lines and let us choose one of two possible directions on each segment (draw an arrow on each segment). Consider a red point R where two given lines a and b meet, and the four segments a_1, a_2, b_1, b_2 with endpoint R (so that $a_i \subset a$, $b_j \subset b$). R is called a *saddle* if on a_1, a_2 the arrows go out of R while on b_1, b_2 the arrows enter R, or visa versa, on b_1, b_2 the arrows go out of R while on a_1, a_2 the arrows enter R. The set of arrows (chosen on all segments) is said to be *good* if all red points are saddles. It is sufficient to prove that there exists a good set of arrows. Indeed, if initially Turbo is moving along (or opposite) the arrow, then this condition holds after she turns at a red point.
The given lines cut the plane into *regions*. Further we need the following *property* of the good set of arrows (this property directly follows from the definition): the boundary of any bounded region is a directed cycle of arrows; the boundary of any unbounded region is a directed chain of arrows.
We construct a good set of arrows by induction on n with trivial base $n = 1$. Now erase one of n given lines and assume we have a good set of arrows for remaining $n-1$ lines. Now restore the n-th line ℓ, assume that ℓ is horizontal. Denote by $A_1, \ldots, A_{n-1}$ all *new* red points on ℓ from the left to the right. Each of A_i belongs to some *old* segment m_i of the line ℓ_i. Let us call A_i *ascending* if the arrow on m_i goes up, and *descending* if the arrow on m_i goes down. Consider the region containing the segment A_iA_{i+1}. By the *property*, A_i and A_{i+1} can not be both ascending or both descending. Thus we can choose arrows on all pieces of ℓ so that each arrow goes from a descending to an ascending vertex.

Each of points A_i cuts m_i into two new pieces; the direction of new pieces supposed to be the same as on m_i. Now simultaneously change the direction of arrows on all pieces below the line ℓ. It is easy to see that $A_1, \ldots, A_{n-1}$ become saddles, while the other red points remain saddles. This completes the induction step.

Problem 4

Let $n \geq 1$ be an integer and let $t_1 < t_2 < \ldots < t_n$ be positive integers. In a group of $t_n + 1$ people, some games of chess are played. Two people can play each other at most once. Prove that it is possible for the following conditions to hold at the same time:

i) The number of games played by each person is one of $t_1, t_2, \ldots, t_n$,

ii) For every i with $1 \leq i \leq n$, there is someone who has played exactly t_i games of chess.

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Comment

In graph theory terms the problem is to prove that for any finite nonempty set $\mathcal{T}$ of positive integers there exists a graph of size $\max \mathcal{T} + 1$ such that the degree set of the graph is equal to $\mathcal{T}$.

Among graph theory specialists a generalization of this problem is known [1]. Nevertheless, the problem still suited the contest.

1. Solution (see also [2])

Let $\mathcal{T} = \{t_1, \ldots, t_n\}$. The proof proceeds by induction on $n = |\mathcal{T}|$. If $n = 1$ and $\mathcal{T} = \{t\}$, choose a group of $t + 1$ people and let every pair of two persons play against each other. Then every person has played t games and the conditions of the problem are satisfied.

In the inductive step, suppose that $\mathcal{T}$ has $n \geq 2$ elements $t_1 < t_2 < \cdots < t_n$. Consider the set

$$\mathcal{T}' = \{t_n - t_{n-1}, t_n - t_{n-2}, \ldots, t_n - t_1\}.$$

By the inductive hypothesis, there exists a group G' of $t_n - t_1 + 1$ people that satisfies the conditions of the problem for T'.

Next construct a group G'' of $t_n + 1$ people by adding t_1 people who do not know any of the other $t_n - t_1 + 1$ people in G'. Finally, construct a group G by complementing the knowledge relation in G'': two persons play against each other in G if and only if they do not play against each other in G''.

By construction $t \in \mathcal{T}$ if and only if there exists a person in G'' that played against exactly $t_n - t$ other people (if $t = t_n$, choose one of the t_1 people added to G'). That person knows $t_n - (t_n - t) = t$ other students in G, completing the proof.

2. Solution

Let $\mathcal{T} = \{t_1, \ldots, t_n\}$. The proof proceeds by induction on $n = |\mathcal{T}|$. If $n = 1$ and $\mathcal{T} = \{t\}$, we choose a group of $t + 1$ people such that everyone plays with everyone else. If $n = 2$ and $\mathcal{T} = \{t_1, t_2\}$ with $t_1 < t_2$, divide the $t_2 + 1$ people into groups A resp. B of size t_1 resp. $t_2 - t_1 + 1$ such that everyone from group A played with everyone else whereas people from group B only played with the people from group A. Then the people from group A resp. B played with exactly t_2 resp. t_1 other people. In the inductive step, suppose that T has $n > 2$ elements $t_1 < \ldots < t_n$. Consider the set

$$\mathcal{T}' = (\mathcal{T} \backslash \{t_1, t_n\}) - t_1 = \{t_{n-1} - t_1, t_{n-1} - t_1, \ldots, t_2 - t_1\}.$$

By the induction hypothesis there exists a group C of $t_{n-1} - t_1 + 1$ people that satisfies the conditions of the problem for T'. Next add groups D resp. E of t_1 resp. $t_n - t_{n-1}$ people such that people from group D played with everyone else whereas people from group E only played with the people from group D. Then the people from group C played with t other people if and only if they played with $t - t_1$ many people among C, i.e. if and only if $t \in \{t_2, \ldots, t_{n-1}\} = \mathcal{T} \backslash \{t_1, t_n\}$. People from group D resp. E played with t_n resp t_1 peoples, which completes the proof.

3. Solution

The proof proceeds by induction on $|t_n|$. If $t_n = 1$ we have $n = 1$ and we can consider two persons that play against each other. Then every player has played 1 game and the conditions of the problem are satisfied. If $t_n > 1$ we distinguish the two cases $t_1 > 1$ and $t_1 = 1$. If $t_1 > 1$ there exists, by the induction hypothesis, a group A of size t_n that satisfies the conditions of the problem for $t_1' = t_1 - 1, \ldots, t_n' = t_n - 1$. Now add a new person to A and let him/her play against everyone from A. The new group will be of size $t_n + 1$ and there exists a person which has played t games if and only if there exists a person that has played $t - 1$ games within A, i.e. if and only if $t \in \{t_1, \ldots, t_n\}$. Hence the conditions of the problem are satisfied.

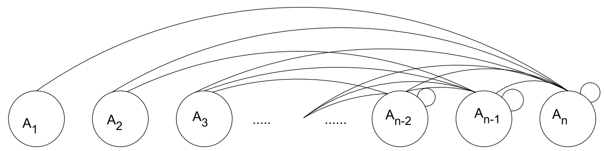
If $t_1 = 1$ there exists, by the induction hypothesis, a group B of size t_{n-1} that satisfies the conditions of the problem for $t_1' = t_2 - 1, \ldots, t_{n-2}' = t_{n-1} - 1$. Now add a new person P and let him/her play with everyone from group B and a group C of size $t_n - t_{n-1} > 0$ and let them play with P. The new group will be of size $t_{n-1} + 1 + (t_n - t_{n-1}) + 1 = t_n + 1$. Since person P has played against everyone he will have played t_n games. The people in C will have played $1 = t_1$ games. There exists a person in B that has played t games if and only if there exist a person in B that has played $t - 1$ games within B, i.e. if and only if $t \in \{t_2, \ldots, t_{n-1}\}$. Hence the conditions of the problem are satisfied.

4. Solution

We generalize the construction for $\mathcal{T} = \{1, ..., n\}$

Construction

Take sets of people $A_1, ..., A_n$. Let all people of A_i play chess with all people in A_j with $j \geq n - i + 1$



Now the number of games played by anyone in A_i is
$(\sum_{j\geq n-i+1}|A_j|)$ or $(\sum_{j\geq n-i+1}|A_j|)-1$ if $i \geq n-i+1$.
Now if we start with one person in each A_i and two people in $A_{\lceil \frac{n}{2} \rceil}$. The number of played games for anyone in A_i is equal to i. In particular this is a construction for $\mathcal{T} = \{1,..,n\}$
Now to get to numbers of general sets $\mathcal{T}$ of size n we can change the sizes of A_i but keep the construction.

Variant 1

Observation 1 *Adding a person to a set A_i increases the number of games played in A_j for $j \geq n-i+1$, by exactly one.*

Start with the construction above and then add $t_1 - 1$ people to group A_n, making the new set of games played equal to $\{t_1, t_1+1, ..., n+t_1-1\}$. Then add t_2-t_1-1 to A_{n-1} to get set of games played to $\{t_1, t_2, t_2+1, ..., n+t_2-2\}$ and repeat until we get to the set $\mathcal{T}$ adding a total of $\sum_{j=1}^{n} t_j - t_{j-1} - 1 = t_n - n$ people (let $t_0 = 0$), so we get $t_n + 1$ people in the end.
Clearly we can start by adding vertices to A_1 or any other set instead of A_n first and obtain an equivalent construction with the same number of people.

Variant 2

It is also possible to calculate the necessary sizes of A_i's all at once. We have by construction the number of games played in A_1 is less than the number of games played in A_2 etc. So we have that in the end we want the games played in A_i to be exactly t_i.
So $(t_t, t_2, t_3, ..., t_n) \overset{!}{=} (|A_n|\ ,\ |A_n|+|A_{n-1}|\ ,\ ...\ ,\ (\sum_{j=2}^{n}|A_j|)-1\ ,\ (\sum_{j=1}^{n}|A_j|)-1)$.
This gives us by induction that $|A_n| \overset{!}{=} t_1$, $|A_{n-1}| \overset{!}{=} t_2 - t_1$, ..., $|A_{\lceil \frac{n}{2} \rceil}| \overset{!}{=} t_{n-\lceil \frac{n}{2} \rceil+1} - t_{n-\lceil \frac{n}{2} \rceil} + 1$, ..., $|A_1| \overset{!}{=} t_n - t_{n-1}$ and a quick calculation shows that the sum of all sets is exactly $1 + \sum_{j=1}^{n}(t_j - t_{j-1}) = t_n + 1$. (where the $+1$ comes from the set $A_{\lceil \frac{n}{2} \rceil}$ and $t_0 = 0$.)

References

[1] Timothy A. Sipka. The orders of graphs with prescribed degree sets. *Journal of Graph Theory*, 4(3):301–307, 1980.

[2] Amitabha Tripathi and Sujith Vijay. A short proof of a theorem on degree sets of graphs. *Discrete Appl. Math.*, 155(5):670–671, 2007.

Problem 5

Let $n \geq 2$ be an integer. An n-tuple $(a_1, a_2, \dots, a_n)$ of positive integers is *expensive* if there exists a positive integer k such that

$$(a_1 + a_2)(a_2 + a_3) \cdot \dots \cdot (a_{n-1} + a_n)(a_n + a_1) = 2^{2k-1}.$$

a) Find all positive integers $n \geq 2$ for which there exists an expensive n-tuple.

b) Prove that for every positive integer m there exists an integer $n \geq 2$ such that m belongs to an expensive n-tuple.

There are exactly n factors in the product on the left hand side.

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Solution 1

a) Notice that for odd integers $n > 2$, the tuple $(1, 1, \dots, 1)$ is expensive. We will prove that there are no expensive n-tuples for even n.

Lemma 0.1. *If an expensive n-tuple exists for some $n \geq 4$, then also an expensive $n - 2$-tuple.*

Proof. In what follows all indices are considered modulo n. Let $(a_1, a_2, \dots, a_n)$ be an expensive n-tuple and a_t the largest element of the tuple. We have the inequalities

$$a_{t-1} + a_t \leq 2a_t < 2(a_t + a_{t+1}) \tag{1}$$
$$a_t + a_{t+1} \leq 2a_t < 2(a_{t-1} + a_t). \tag{2}$$

Since both $a_{t-1} + a_t$ and $a_t + a_{t+1}$ are powers of 2 (they are divisors of a power of 2), we deduce from (1) and (2)

$$a_{t-1} + a_t = a_t + a_{t+1} = 2^r$$

for some positive integer r, and in particular $a_{t-1} = a_{t+1}$.
Consider now the $n-2$-tuple $(b_1, \dots, b_{n-2})$ obtained by removing a_t and a_{t+1} from $(a_1, a_2, \dots, a_n)$. By what we just said we have

$$\prod_{i=1}^{n-2}(b_i + b_{i+1}) = \frac{\prod_{i=1}^{n}(a_i + a_{i+1})}{(a_{t-1} + a_t)(a_t + a_{t+1})} = 2^{2(k-r)-1},$$

and hence $(b_1, \dots, b_{n-2})$ is again expensive. □

From the lemma we now conclude that if there exists an expensive n-tuple for some even n, then also an expensive 2-tuple i.e.

$$(a_1 + a_2)^2 = 2^{2k-1}$$

for some positive integers a_1, a_2, which is impossible since the right hand side is not a square.

b) We prove this by induction. In *a*) we saw that 1 belongs to an expensive n-tuple. Assume now that all odd positive integers less that 2^k belong to an expensive n-tuple, for some $k \geq 1$. Hence for any odd $r < 2^k$ there is an integer n and an expensive n-tuple $(a_1, \dots, r, \dots, a_n)$. We notice that then also $(a_1, \dots, r, 2^{k+1} - r, r, \dots, a_n)$ is expensive. Since $2^{k+1} - r$ can take all odd values between 2^k and 2^{k+1} the induction step is complete.

Solution 2

a) For odd n the tuple $(1, 1, \dots, 1)$ is a solution.
Now consider n even. Since the product $\prod(a_i + a_{i+1})$ is a power of two, every factor needs to be a power of two. We are going to prove that for all tuples $(a_1, \dots, a_n)$ such that $a_i + a_{i+1}$ is always a power of two , it is the case that $\prod(a_i + a_{i+1})$ is equal to an even power of two. We are going to prove this with strong induction on $\sum a_i$. When all a_i are equal to one this is certainly the case. Since $a_i + a_{i+1} > 1$ it is even and we conclude that the a_i are either all odd or all even. In the case they are all even, then consider the tuple $(b_1, \dots, b_n)$ with $b_i = a_i/2$. This tuple clearly satisfies the hypothesis as well and we have $\sum b_i < \sum a_i$. Furthermore we have $\prod(a_i + a_{i+1}) = 2^n \prod(b_i + b_{i+1})$ and since n is even we are done in this case.
Now all a_i are odd. Suppose none of are one, then consider the tuple $(b_1, \dots, b_n)$ with $b_i = (a_i + (-1)^i)/2$. Since all a_i are odd and strictly larger than one, the b_i are positive integers and satisfy $b_i + b_{i+1} = (a_i + a_{i+1})/2$, a power of two. Since $\sum b_i < \sum a_i$ and $\prod(a_i + a_{i+1}) = 2^n \prod(b_i + b_{i+1})$ we are done in this case again. Now there is at least one a_i being one. We may assume $i = 1$, because the condition is cyclic. Moreover we may also assume that $a_2 > 1$ since not all of the a_i are equal to one. Let now k be the smallest index larger than one such that a_k is equal to one. We are not excluding the case $k = n + 1$, yet. Now for $i = 1, \dots, k - 1$ we have $a_i + a_{i+1} > 2$ and thus divisible by four. By induction it easily follows that $a_i \equiv (-1)^{i+1} \mod (4)$ for $i = 1, \dots, k - 1$. In particular, since $a_k = 1$ we find that k is odd and at least three. Now consider the tuple $(b_1, \dots, b_n)$ with $b_i = (a_i - (-1)^i)/2$ for $i = 1, \dots, k$ and $b_i = a_i$ otherwise. This is again a tuple that satisfies the hypothesis, since $b_1 = a_1 = 1 = b_k = a_k$. Moreover $b_2 < a_2$ and thus $\sum b_i < \sum a_i$. Finally we have $\prod(a_i + a_{i+1}) = 2^{k-1} \prod(b_i + b_{i+1})$ and since k is odd we conclude the proof.

b) We use some of the ideas from a). Consider the operators $T_\pm(n) = 2n \pm 1$. We claim that for every odd integer m there is an integer r and signs $\epsilon_i \in \{+, -\}$ for $i = 1, \dots r$ such that $T_{\epsilon_r} \circ \cdots \circ T_{\epsilon_1}(1) = m$. This is certainly true for $m = 1$ and for $m > 1$ we find that $m = T_-((m + 1)/2)$ if $m \equiv 1 \mod (4)$ and $m = T_+((m - 1)/2)$ if $m \equiv 3 \mod (4)$. Note that both $(m + 1)/2$ and $(m - 1)/2$ are odd integers in their respective cases and $(m-1)/2 \leq (m+1)/2 < m$ for $m > 1$. Therefore iterating the procedure will eventually terminate in one.
For the construction it is most convenient to set $n = 2l + 1$ and label the tuple $(a_{-l}, a_{-l+1}, \dots, a_l)$. For $m = 1$ we have the expensive tuple $(1, 1, 1)$. For $m > 1$ we will define operators $T_\pm$ on expensive tuples with the condition $a_{-l} = a_l = 1$ that give rise to a new expensive tuple $(b_{-l'}, \dots, b_{l'})$ with $b_{-l'} = b_{l'} = 1$ and $b_0 = T_\pm a_0$. It is then clear that $T_{\epsilon_r} \circ \cdots \circ T_{\epsilon_1}((1, 1, 1))$ is an expensive tuple containing m. We define $T_\pm$ as follows: set $l' = l + 1$ and $b_{-l'} = b_{l'} = 1$ and $b_i = T_{\pm(-1)^i}(a_i)$ for $i = -l, \dots, l$. Here we identify $+$ with $+1$ and $-$ with -1. We are left to prove that the new tuple is indeed expensive. If $\pm(-1)^l = -1$, then $\prod(b_i + b_{i+1}) = 4 \cdot 2^{2l} \prod(a_i + a_{i+1})$, and if $\pm(-1)^l = +1$, then $\prod(b_i + b_{i+1}) = 4 \cdot 2^{2l+2} \prod(a_i + a_{i+1})$. In both cases we end up with an expensive tuple again.

Problem 6

Let ABC be an acute-angled triangle in which no two sides have the same length. The reflections of the centroid G and the circumcentre O of ABC in its sides BC, CA, AB are denoted by G_1, G_2, G_3, and O_1, O_2, O_3, respectively. Show that the circumcircles of the triangles G_1G_2C, G_1G_3B, G_2G_3A, O_1O_2C, O_1O_3B, O_2O_3A and ABC have a common point.

The centroid of a triangle is the intersection point of the three medians. A median is a line connecting a vertex of the triangle to the midpoint of the opposite side.

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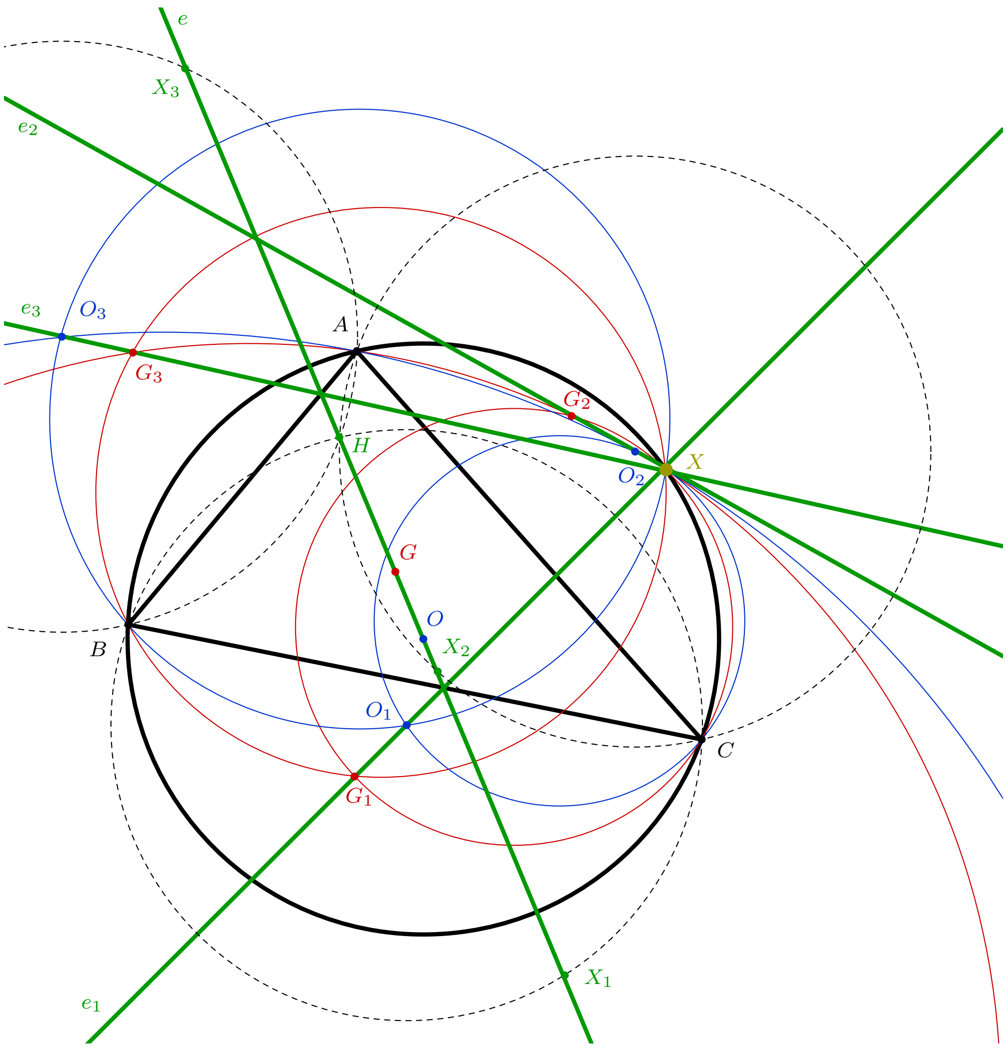
Solution 1 (Euler lines)

Let H denote the orthocenter of ABC, and let e denote its Euler line. Let e_1, e_2, e_3 denote the respective reflections of e in BC, CA, AB. The proof naturally divides into two parts: we first show that pairwise intersections of the circles in question correspond to pairwise intersections of e_1, e_2, e_3, and then prove that e_1, e_2, e_3 intersect in a single point on the circumcircle of ABC.

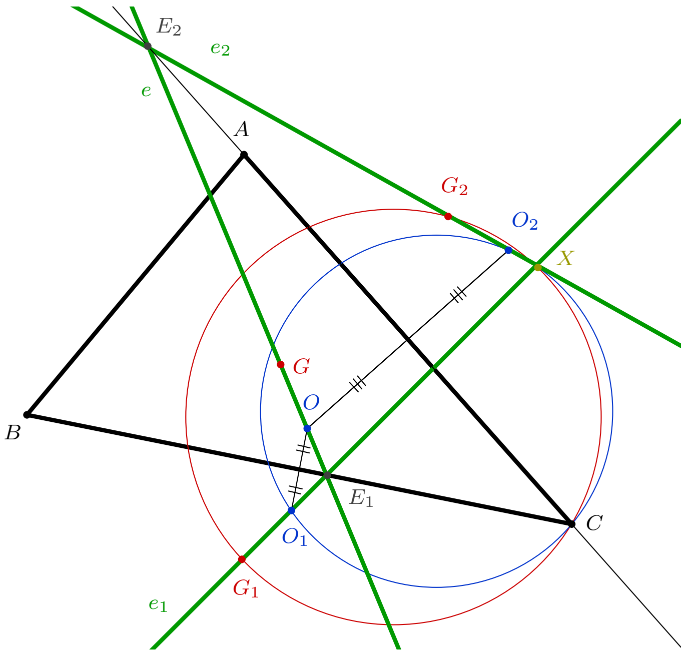


Now consider for example the circumcircles of O_1O_2C and G_1G_2C. By construction, it is clear that $\angle O_2CO_1 = \angle G_2CG_1 = 2\angle ACB$. Let G_1O_1 and G_2O_2 meet at X, and let e meet e_1, e_2 at E_1, E_2, respectively, as shown in the diagram below. Chasing angles,

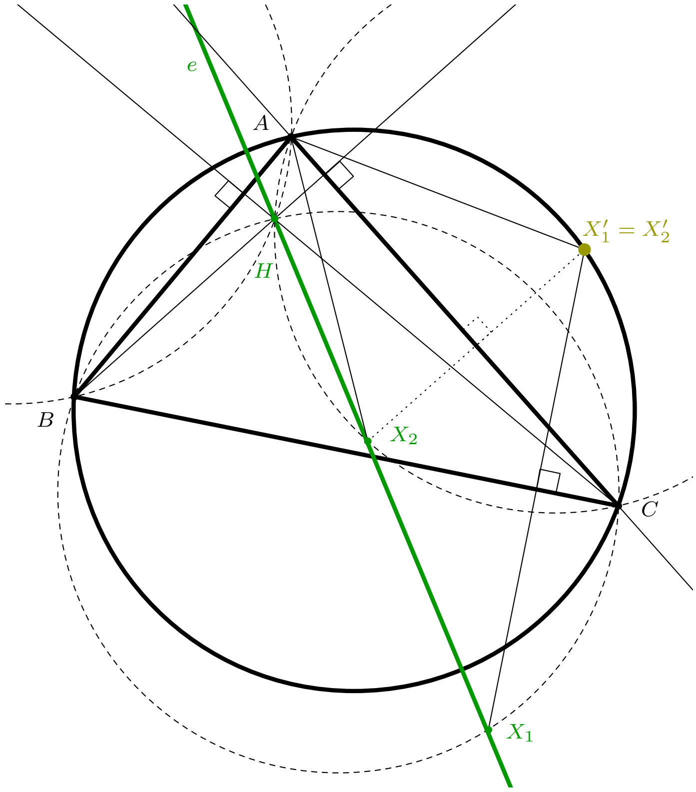
$$\begin{aligned}\angle G_2XG_1 = \angle O_2XO_1 = \angle E_2XE_1 &= 180^\circ - \angle E_1E_2X - \angle XE_1E_2 \\ &= 180^\circ - 2\angle E_1E_2C - \big(\angle CE_1E_2 - \angle CE_1X\big).\end{aligned}$$

But $\angle CE_1X = \angle BE_1O_1 = \angle BE_1O = 180^\circ - \angle CE_1E_2$, and thus

$$\angle G_2XG_1 = \angle O_2XO_1 = 2\big(180^\circ - \angle E_1E_2C - \angle CE_1E_2\big) = 2\angle ACB.$$



It follows from this that X lies on the circumcircles of G_1G_2C and O_1O_2C. In other words, the second point of intersection of the circumcircles of G_1G_2C and O_1O_2C is the intersection of e_1 and e_2. Similarly, the circumcircles of G_1G_3B and O_1O_3B meet again at the intersection of e_1 and e_3, and those of G_2G_3A and O_2O_3A meet again at the intersection of e_2 and e_3.



It thus remains to show that e_1, e_2, e_3 are concurrent, and intersect on the circumcircle of ABC. Let e meet the circumcircles of the triangles BCH, ACH, ABH at X_1, X_2, X_3, respectively. It is well known that the reflections of H in the sides of ABC lie on the circumcircle of ABC. For this reason, the circumcircles of BCH, ACH, ABH have the same radius as the circumcircle of ABC, and hence the reflections X_1', X_2', X_3' of X_1, X_2, X_3 in the sides $[BC], [CA], [AB]$ lie on the circumcircle of ABC. By definition, X_1', X_2', X_3' lie on e_1, e_2, e_3, respectively. It thus remains to show that they coincide.
To show that, for example, $X_1' = X_2'$, it will be sufficient to show that $\angle X_2AC = \angle X_1'AC$, since we have already shown that X_1' and X_2' lie on the circumcircle of ABC. But, chasing angles in the diagram above,

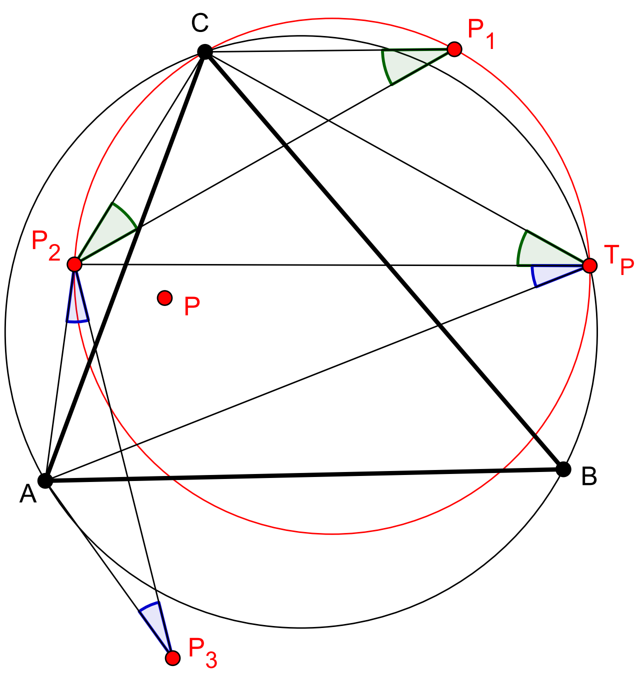
$$\begin{aligned}\angle X_1'AC &= \angle X_1'AB - \angle BAC = \left(180^\circ - \angle X_1'CB\right) - \angle BAC \\ &= 180^\circ - \angle X_1CB - \angle BAC = \left(180^\circ - \angle BAC\right) - \angle X_1HB \\ &= \angle BHC - \angle X_1HB = \angle X_2HC = \angle X_2AC,\end{aligned}$$

where we have used the fact that $BHCX_1$ and AHX_2C are cyclic by construction, and the fact that $\angle BHC = 180^\circ - \angle BAC$. This shows that $X_1' = X_2'$. Similarly, $X_1' = X_3'$, which completes the proof. $\square$

Remark. The statement of the problem remains true if O and G are replaced with two points that are aligned with the orthocenter H of the triangle, and indeed, the proof above did not require any property of G and O other than the fact that they lie on a line through H, the Euler line.

Solution 2

The proof consists of two parts. First, we show that if P is any point inside the triangle ABC and P_1, P_2, P_3 are its reflections in the sides BC, CA, AB, then the circumcircles of the triangles P_1P_2C, P_1P_3B, P_2P_3A intersect in a point T_P on the circumcircle of the triangle ABC. In the second part, we show that T_G coincides with T_O.
Now let P be any point inside the triangle ABC and let P_1, P_2, P_3 be the reflections in the sides as above. Let T_P be the second intersection of the circumcircles of the triangles P_1P_2C and ABC. We want to show that T_P lies on the circumcircles of the triangles P_1P_3B and P_2P_3A.



By construction, we have $P_1C = P_2C$, hence

$$\angle CP_1P_2 = 90^\circ - \frac{1}{2}\angle P_2CP_1 = 90^\circ - \angle ACB.$$

Similarly, $\angle P_2P_3A = 90^\circ - \angle BAC$. This gives us

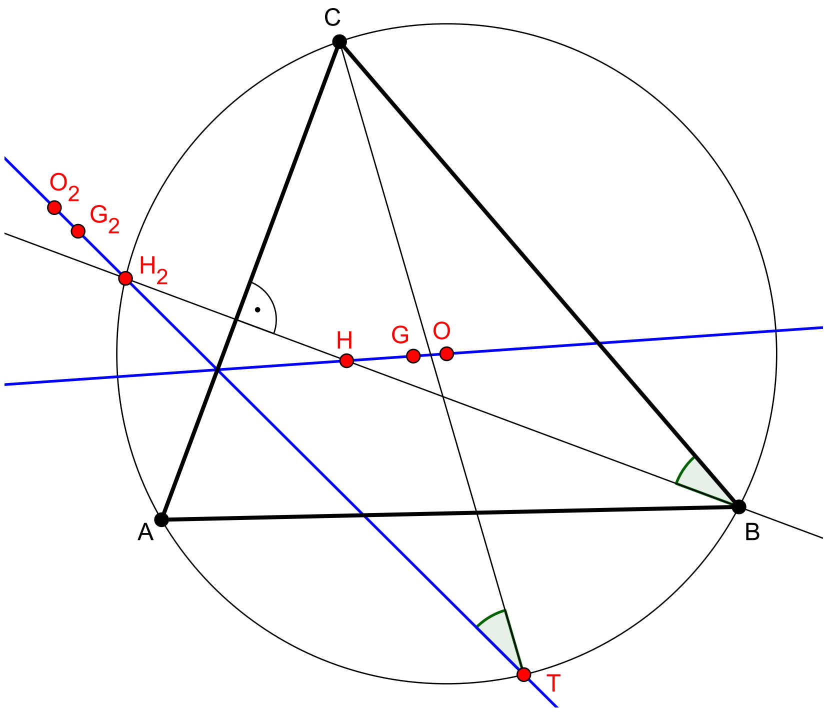
$$\begin{aligned}\angle P_2T_PA &= \angle CT_PA - \angle CT_PP_2 = \angle CBA - \angle CP_1P_2 \\ &= \angle CBA - 90^\circ + \angle ACB = 90^\circ - \angle BAC = \angle P_2P_3A,\end{aligned}$$

so T_P lies on the circumcircle of the triangle P_2P_3A. Similarly, T_P lies on the circumcircle of the triangle P_1P_3B which completes the first part.
Note that if P_2 is given, then T_P is the unique point on the circumcircle of the triangle ABC with $\angle CT_PP_2 = 90^\circ - \angle ACB$. In the second part, we will use this as follows: If we can find a point T on the circumcircle of the triangle ABC with $\angle CTG_2 = \angle CTO_2 = 90^\circ - \angle ACB$,

then $T = T_G = T_O$ and we are done.
Let H be the orthocenter of the triangle ABC and let H_2 be the reflection in the side AC. It is known that H_2 lies on the circumcircle of the triangle ABC. G, O, H lie on the Euler line, so G_2, O_2, H_2 are collinear as well. Let T be the second intersection of G_2H_2 and the circumcircle of the triangle ABC. We can now complete the proof by seeing that

$$\angle CTG_2 = \angle CTO_2 = \angle CTH_2 = \angle CBH_2 = 90^\circ - \angle ACB.$$

□



Solution 3 (complex numbers)

For every point P, let p denote the corresponding complex number. Set O to be the origin, so $o = 0$, and without loss of generality we can assume that a, b and c lie on the unit circle. Then the centroid can be expressed as $g = \frac{a+b+c}{3}$.
The segments oo_1 and bc have a common midpoint, so $o_1 + o = b + c$, and then $o_1 = b + c$. Similarly $o_2 = a + c$ and $o_3 = a + b$. In order to compute g_1, define y to be the projection of g onto bc. Since b and c are on the unit circle, it is well known that y can be expressed as

$$y = \frac{1}{2}(b + c + g - bc\bar{g}).$$

By using $\bar{a} = \frac{1}{a}$, $\bar{b} = \frac{1}{b}$ and $\bar{c} = \frac{1}{c}$ (points on the unit circle), we obtain

$$g_1 = b + c - \frac{ab + bc + ca}{3a}.$$

Similarly, we get $g_2 = a + c - \frac{ab+bc+ca}{3b}$ and $g_3 = a + b - \frac{ab+bc+ca}{3c}$.
1) Proof that circumcircles of triangles abc, o_1o_2c, o_1o_3b and o_2o_3a have common point.
Let x be the point of intersection of circumcircles of triangles o_1o_2c and abc ($x \neq c$). We know that x, o_1, o_2 and c are concyclic if and only if $\frac{x-c}{o_1-c} : \frac{x-o_2}{o_1-o_2}$ is real number, which is equivalent to

$$\frac{x-c}{\bar{x}-\bar{c}} \cdot \frac{o_1-o_2}{\bar{o_1}-\bar{o_2}} = \frac{o_1-c}{\bar{o_1}-\bar{c}} \cdot \frac{x-o_2}{\bar{x}-\bar{o_2}}. \tag{1}$$

Since x and c are on the unit circle $\frac{x-c}{\bar{x}-\bar{c}} = -xc$. Also, $\frac{o_1-o_2}{\bar{o_1}-\bar{o_2}} = \frac{b-a}{\bar{b}-\bar{a}} = -ab$, and $\frac{o_1-c}{\bar{o_1}-\bar{c}} = \frac{b}{\bar{b}} = b^2$. Since $\bar{x} = \frac{1}{x}$, from (1) and previous relations, we have:

$$x = \frac{ab + bc + ca}{a + b + c}.$$

This formula is symmetric, so we conclude that x also belongs to circumcircles of o_1o_3b and o_2o_3a.
2) Proof that x belongs to circumcircles of g_1g_2c, g_1g_3b and g_2g_3a.
Because of symmetry, it is enough to prove that x belongs to circumcircle of g_1g_2c, i.e. to prove the following:

$$\frac{x-c}{\bar{x}-\bar{c}} \cdot \frac{g_1-g_2}{\bar{g_1}-\bar{g_2}} = \frac{g_1-c}{\bar{g_1}-\bar{c}} \cdot \frac{x-g_2}{\bar{x}-\bar{g_2}}. \tag{2}$$

Easy computations give that

$$g_1 - g_2 = (b-a)\frac{2ab - bc - ac}{3ab}, \quad \bar{g_1} - \bar{g_2} = (\bar{b} - \bar{a})\frac{2 - \frac{a}{c} - \frac{b}{c}}{3},$$

and then

$$\frac{g_1 - g_2}{\bar{g_1} - \bar{g_2}} = \frac{c(bc + ac - 2ab)}{2c - a - b}.$$

On the other hand we have

$$g_1 - c = \frac{2ab - bc - ac}{3a}, \quad \bar{g_1} - \bar{c} = \frac{2c - a - b}{3bc}.$$

This implies

$$\frac{g_1 - c}{\bar{g_1} - \bar{c}} = \frac{2ab - bc - ac}{2c - a - b} \cdot \frac{bc}{a}.$$

Then (2) is equivalent to

$$-xc \cdot \frac{c(bc + ac - 2ab)}{2c - a - b} = \frac{2ab - bc - ac}{2c - a - b} \cdot \frac{bc}{a} \cdot \frac{x - g_2}{\bar{x} - \bar{g_2}}$$
$$\iff xca(\bar{x} - \bar{g_2}) = b(x - g_2),$$

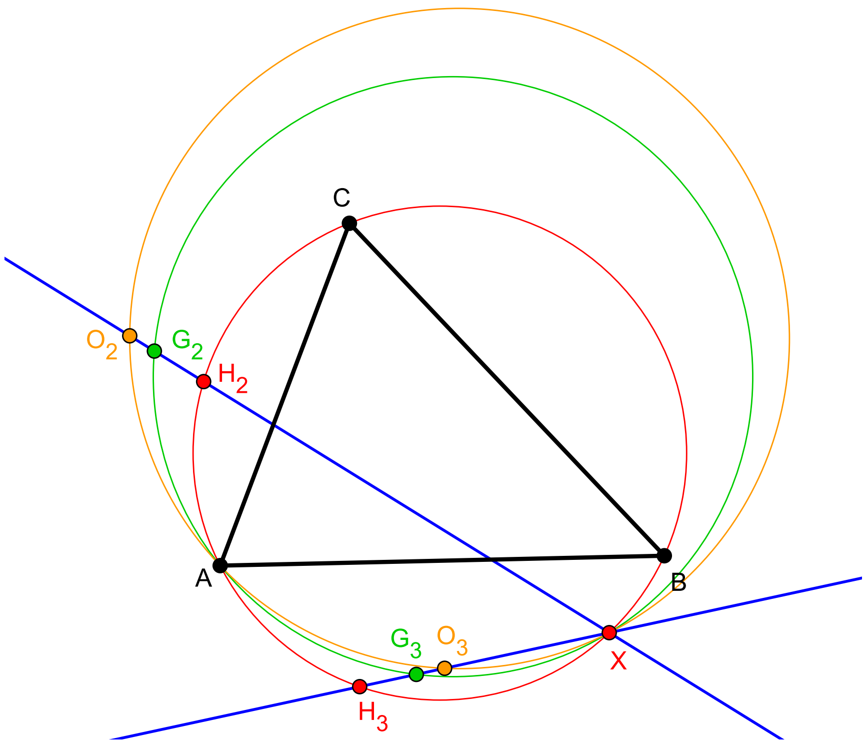
which is also equivalent to

$$\frac{ab + bc + ca}{a + b + c} \cdot ca \left(\frac{a + b + c}{ab + bc + ca} - \frac{1}{a} - \frac{1}{c} + \frac{a + b + c}{3ac} \right) = b \cdot \left(\frac{ab + bc + ca}{a + b + c} - a - c + \frac{ab + bc + ca}{3b} \right).$$

The last equality can easily be verified, which implies that x belongs to circumcircle of triangle $g_1 g_2 c$. This concludes our proof. $\square$

Solution 4 (rotation)

The first part of the first solution and the second part of the second solution can also be done by the following rotation argument: The rotation through $2\angle BAC$ about A takes O_3 to O_2, G_3 to G_2 and H_3 to H_2 (again, H_2, H_3 are the reflections of the orthocenter H in the sides CA, AB). Let X be the intersection of the Euler line reflections e_2 (going through O_2, G_2, H_2) and e_3 (going through O_3, G_3, H_3). We now use the well-known fact that if a rotation about a point A takes a line l and a point P on l to the line l' and the point P', then the quadrilateral $APP'X$ is cyclic, where X is the intersection of l and l'. For this reason, AO_3O_2X, AG_3G_2X and AH_3H_2X are cyclic quadrilaterals. Since H_2 and H_3 lie on the circumcircle of the triangle ABC, the circumcircles of the triangles AH_3H_2 and ABC are the same, hence X lies on the circumcircle of ABC.
This proves the first part of the first solution as well as the second part of the second solution. $\square$



Remark: Problem 6 is a special case of Corollary 3 in Darij Grinberg's paper *Anti-Steiner points with respect to a triangle.*