Sunday, April 9, 2017

Problem 4. Let $n \geq 1$ be an integer and let $t_1 < t_2 < \ldots < t_n$ be positive integers. In a group of $t_n + 1$ people, some games of chess are played. Two people can play each other at most once. Prove that it is possible for the following two conditions to hold at the same time:

(i) The number of games played by each person is one of $t_1, t_2, \ldots, t_n$.

(ii) For every i with $1 \leq i \leq n$, there is someone who has played exactly t_i games of chess.

Problem 5. Let $n \geq 2$ be an integer. An n-tuple $(a_1, a_2, \ldots, a_n)$ of not necessarily different positive integers is *expensive* if there exists a positive integer k such that

$$(a_1 + a_2)(a_2 + a_3) \cdot \ldots \cdot (a_{n-1} + a_n)(a_n + a_1) = 2^{2k-1}.$$

a) Find all integers $n \geq 2$ for which there exists an expensive n-tuple.

b) Prove that for every odd positive integer m there exists an integer $n \geq 2$ such that m belongs to an expensive n-tuple.

There are exactly n factors in the product on the left hand side.

Problem 6. Let ABC be an acute-angled triangle in which no two sides have the same length. The reflections of the centroid G and the circumcentre O of ABC in its sides BC, CA, AB are denoted by G_1, G_2, G_3, and O_1, O_2, O_3, respectively. Show that the circumcircles of the triangles G_1G_2C, G_1G_3B, G_2G_3A, O_1O_2C, O_1O_3B, O_2O_3A and ABC have a common point.

The centroid of a triangle is the intersection point of the three medians. A median is a line connecting a vertex of the triangle to the midpoint of the opposite side.