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PROBLEMS

Day 1

Problem 1. Let $\triangle ABC$ be an acute-angled triangle, and let D be the foot of the altitude from C. The angle bisector of $\angle ABC$ intersects CD at E and meets the circumcircle ω of triangle $\triangle ADE$ again at F. If $\angle ADF = 45^\circ$, show that CF is tangent to ω.

Problem 2. A *domino* is a 2×1 or 1×2 tile. Determine in how many ways exactly n^2 dominoes can be placed without overlapping on a $2n \times 2n$ chessboard so that every 2×2 square contains at least two uncovered unit squares which lie in the same row or column.

Problem 3. Let n, m be integers greater than 1, and let $a_1, a_2, \ldots, a_m$ be positive integers not greater than n^m. Prove that there exist positive integers $b_1, b_2, \ldots, b_m$ not greater than n, such that

$$\gcd(a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) < n,$$

where $\gcd(x_1, x_2, \ldots, x_m)$ denotes the greatest common divisor of $x_1, x_2, \ldots, x_m$.

Day 2

Problem 4. Determine whether there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers which satisfies the equality

$$a_{n+2} = a_{n+1} + \sqrt{a_{n+1} + a_n}$$

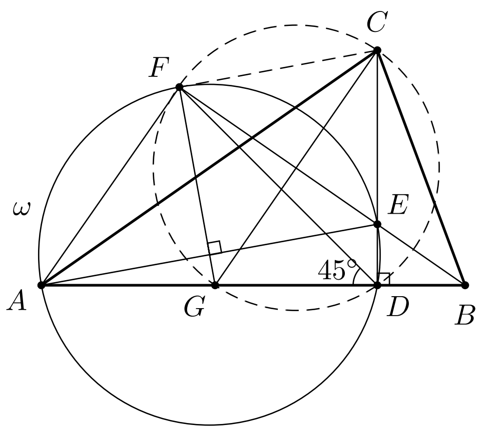
for every positive integer n.

Problem 5. Let m, n be positive integers with $m > 1$. Anastasia partitions the integers $1, 2, \ldots, 2m$ into m pairs. Boris then chooses one integer from each pair and finds the sum of these chosen integers. Prove that Anastasia can select the pairs so that Boris cannot make his sum equal to n.

Problem 6. Let H be the orthocentre and G be the centroid of acute-angled triangle $\triangle ABC$ with $AB \neq AC$. The line AG intersects the circumcircle of $\triangle ABC$ at A and P. Let P' be the reflection of P in the line BC. Prove that $\angle CAB = 60^\circ$ if and only if $HG = GP'$.

Problem 1. *Let $\triangle ABC$ be an acute-angled triangle, and let D be the foot of the altitude from C. The angle bisector of $\angle ABC$ intersects CD at E and meets the circumcircle ω of triangle $\triangle ADE$ again at F. If $\angle ADF = 45°$, show that CF is tangent to ω.*

(Luxembourg)



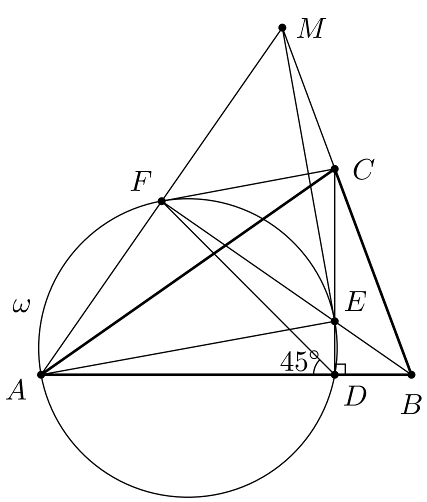
Solution 1: Since $\angle CDF = 90° - 45° = 45°$, the line DF bisects $\angle CDA$, and so F lies on the perpendicular bisector of segment AE, which meets AB at G. Let $\angle ABC = 2\beta$. Since $ADEF$ is cyclic, $\angle AFE = 90°$, and hence $\angle FAE = 45°$. Further, as BF bisects $\angle ABC$, we have $\angle FAB = 90° - \beta$, and thus

$$\angle EAB = \angle AEG = 45° - \beta, \quad \text{and} \quad \angle AED = 45° + \beta,$$

so $\angle GED = 2\beta$. This implies that right-angled triangles $\triangle EDG$ and $\triangle BDC$ are similar, and so we have $|GD|/|CD| = |DE|/|DB|$. Thus the right-angled triangles $\triangle DEB$ and $\triangle DGC$ are similar, whence $\angle GCD = \angle DBE = \beta$. But $\angle DFE = \angle DAE = 45° - \beta$, then $\angle GFD = 45° - \angle DFE = \beta$. Hence $GDCF$ is cyclic, so $\angle GFC = 90°$, whence CF is perpendicular to the radius FG of ω. It follows that CF is a tangent to ω, as required.

Solution 2: As $\angle ADF = 45°$ line DF is an exterior bisector of $\angle CDB$. Since BF bisects $\angle DBC$ line CF is an exterior bisector of $\angle BCD$. Let $\angle ABC = 2\beta$, so $\angle ECF = (\angle DBC + \angle CDB)/2 = 45° + \beta$. Hence $\angle CFE = 180° - \angle ECF - \angle BCE - \angle EBC = 180° - (45° + \beta + 90° - 2\beta + \beta) = 45°$. It follows that $\angle FDC = \angle CFE$, then CF is tangent to ω.

Solution 3: Note that AE is diameter of circumcircle of $\triangle ABC$ since $\angle CDF = 90°$. From $\angle AEF = \angle ADF = 45°$ it follows that triangle $\triangle AFE$ is right-angled and isosceles. Without loss of generality, let points A, E and F have coordinates $(-1, 0)$, $(1, 0)$ and $(0, 1)$ respectively. Points F, E, B are collinear, hence B have coordinates $(b, 1-b)$ for some $b \neq -1$. Let point C' be intersection of line tangent to circumcircle of $\triangle AFE$ at F with line ED. Thus C' have coordinates $(c, 1)$ and from $\overline{C'E} \perp \overline{AB}$ we get $c = 2b/(b+1)$. Now vector $\overline{BC'} = (2b/(b+1) - b, b) = b/(b+1) \cdot (1-b, b+1)$, vector $\overline{BF} = (-b, b) = (-1, 1) \cdot b$ and vector $\overline{BA} = (-(b+1), -(1-b))$. Its clear that $(1-b, b+1)$ and $(-(b+1), -(1-b))$ are symmetric with respect to $\overline{FE} = (-1, 1)$, hence BF bisects $\angle C'BA$ and $C' = C$ which completes the proof.



Solution 4: Again F lies on the perpendicular bisector of segment AE, so $\triangle AFE$ is right-angled and isosceles. Let M be an intersection of BC and AF. Note that $\triangle AMB$ is isosceles since BF is a bisector and altitude in this triangle. Thus BF is a symmetry line of $\triangle AMB$. Then $\angle FDA = \angle FEA = \angle MEF = 45°$, $AF = FE = FM$ and $\angle DAE = \angle EMC$. Let us show that $EC = CM$. Indeed,

$$\angle CEM = 180° - (\angle AED + \angle FEA + \angle MEF) = 90° - \angle AED =$$
$$= \angle DAE = \angle EMC.$$

It follows that $FMCE$ is a kite, since $EF = FM$ and $MC = CE$. Hence $\angle EFC = \angle CFM = \angle EDF = 45°$, so FC is tangent to ω.

Solution 5: Let the tangent to ω at F intersect CD at C'. Let $\angle ABF = \angle FBC = \beta$. It follows that $\angle C'FE = 45°$ since $C'F$ is tangent. We have

$$\frac{\sin\angle BDC}{\sin\angle CDF} \cdot \frac{\sin\angle DFC'}{\sin\angle C'FB} \cdot \frac{\sin\angle FBC}{\sin\angle CBD} = \frac{\sin 90°}{\sin 45°} \cdot \frac{\sin(90° - \beta)}{\sin 45°} \cdot \frac{\sin\beta}{\sin 2\beta} = \frac{2\sin\beta\cos\beta}{\sin 2\beta} = 1.$$

So by trig Ceva on triangle $\triangle BDF$, lines FC', DC and BC are concurrent (at C), so $C = C'$. Hence CF is tangent to ω.

Problem 2. *A domino is a 2×1 or 1×2 tile. Determine in how many ways exactly n^2 dominoes can be placed without overlapping on a $2n \times 2n$ chessboard so that every 2×2 square contains at least two uncovered unit squares which lie in the same row or column.*

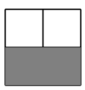
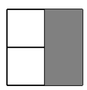
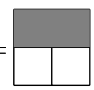
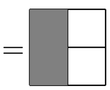
(Turkey)

Solution: The answer is $\binom{2n}{n}^2$.

Divide the schessboard into 2×2 squares. There are exactly n^2 such squares on the chessboard. Each of these squares can have at most two unit squares covered by the dominos. As the dominos cover exactly $2n^2$ squares, each of them must have exactly two unit squares which are covered, and these squares must lie in the same row or column.

We claim that these two unit squares are covered by the same domino tile. Suppose that this is not the case for some 2×2 square and one of the tiles covering one of its unit squares sticks out to the left. Then considering one of the leftmost 2×2 squares in this division with this property gives a contradiction.

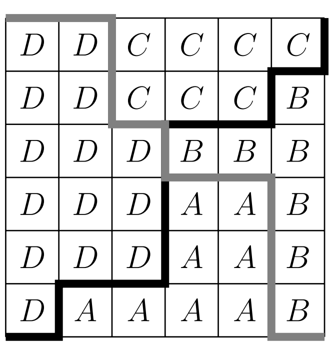
Now consider this $n \times n$ chessboard consisting of 2×2 squares of the original board. Define A, B, C, D as the following configurations on the original chessboard, where the gray squares indicate the domino tile, and consider the covering this $n \times n$ chessboard with the letters A, B, C, D in such a

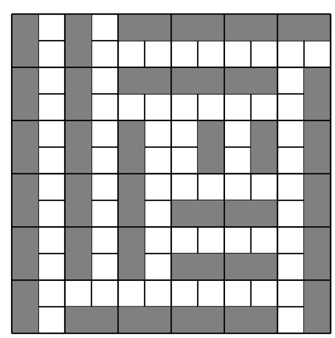
$A =$  $B =$  $C =$  $D =$ 

way that the resulting configuration on the original chessboard satisfies the condition of the question.

Note that then a square below or to the right of one containing an A or B must also contain an A or B. Therefore the (possibly empty) region consisting of all squares containing an A or B abuts the lower right corner of the chessboard and is separated from the (possibly empty) region consisting of all squares containing a C or D by a path which goes from the lower left corner to the upper right corner of this chessboard and which moves up or right at each step.

A similar reasoning shows that the (possibly empty) region consisting of all squares containing an A or D abuts the lower left corner of the chessboard and is separated from the (possibly empty) region consisting of all squares containing a B or C by a path which goes from the upper left corner to the lower right corner of this chessboard and which moves down or right at each step.





Therefore the $n \times n$ chessboard is divided by these two paths into four (possibly empty) regions that consist respectively of all squares containing A or B or C or D. Conversely, choosing two such paths and filling the four regions separated by them with As, Bs, Cs and Ds counterclockwise starting at the bottom results in a placement of the dominos on the original board satisfying the condition of the question.

As each of these paths can be chosen in $\binom{2n}{n}$ ways, there are $\binom{2n}{n}^2$ ways the dominos can be placed.

Problem 3. *Let n, m be integers greater than 1, and let $a_1, a_2, \ldots, a_m$ be positive integers not greater than n^m. Prove that there exist positive integers $b_1, b_2, \ldots, b_m$ not greater than n, such that*

$$\gcd(a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) < n,$$

where $\gcd(x_1, x_2, \ldots, x_m)$ denotes the greatest common divisor of $x_1, x_2, \ldots, x_m$.

(USA)

Solution 1: Suppose without loss of generality that a_1 is the smallest of the a_i. If $a_1 \geq n^m - 1$, then the problem is simple: either all the a_i are equal, or $a_1 = n^m - 1$ and $a_j = n^m$ for some j. In the first case, we can take (say) $b_1 = 1$, $b_2 = 2$, and the rest of the b_i can be arbitrary, and we have

$$\gcd(a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) \leq \gcd(a_1 + b_1, a_2 + b_2) = 1.$$

In the second case, we can take $b_1 = 1$, $b_j = 1$, and the rest of the b_i arbitrary, and again

$$\gcd(a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) \leq \gcd(a_1 + b_1, a_j + b_j) = 1.$$

So from now on we can suppose that $a_1 \leq n^m - 2$.

Now, let us suppose the desired $b_1, \ldots, b_m$ do not exist, and seek a contradiction. Then, for any choice of $b_1, \ldots, b_m \in \{1, \ldots, n\}$, we have

$$\gcd(a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) \geq n.$$

Also, we have

$$\gcd(a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) \leq a_1 + b_1 \leq n^m + n - 2.$$

Thus there are at most $n^m - 1$ possible values for the greatest common divisor. However, there are n^m choices for the m-tuple $(b_1, \ldots, b_m)$. Then, by the pigeonhole principle, there are two m-tuples that yield the same values for the greatest common divisor, say d. But since $d \geq n$, for each i there can be at most one choice of $b_i \in \{1, 2, \ldots, n\}$ such that $a_i + b_i$ is divisible by d — and therefore there can be at most one m-tuple $(b_1, b_2, \ldots, b_m)$ yielding d as the greatest common divisor. This is the desired contradiction.

Solution 2: Similarly to Solution 1 suppose that $a_1 \leq n^m - 2$. The gcd of $a_1+1, a_2+1, a_3+1, \ldots, a_m+1$ is co-prime with the gcd of $a_1 + 1, a_2 + 2, a_3 + 1, \ldots, a_m + 1$, thus $a_1 + 1 \geq n^2$. Now change another 1 into 2 and so on. After $m - 1$ changes we get $a_1 + 1 \geq n^m$ which gives us a contradiction.

Solution 3: We will prove stronger version of this problem:

For $m, n > 1$, let $a_1, \ldots, a_m$ be positive integers with at least one $a_i \leq n^{2^{m-1}}$. Then there are integers $b_1, \ldots, b_m$, each equal to 1 or 2, such that $\gcd(a_1 + b_1, \ldots, a_m + b_m) < n$.

Proof: Suppose otherwise. Then the 2^{m-1} integers $\gcd(a_1 + b_1, \ldots, a_m + b_m)$ with $b_1 = 1$ and $b_i = 1$ or 2 for $i > 1$ are all pairwise coprime, since for any two of them, there is some $i > 1$ with $a_i + 1$ appearing in one and $a_i + 2$ in the other. Since each of these 2^{m-1} integers divides $a_1 + 1$, and each is $\geq n$ with at most one equal to n, it follows that $a_1 + 1 \geq n(n+1)^{2^{m-1}-1}$ so $a_1 \geq n^{2^{m-1}}$. The same is true for each a_i, $i = 1, \ldots, n$, a contradiction.

Remark: Clearly the $n^{2^{m-1}}$ bound can be strengthened as well.

Problem 4. *Determine whether there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers which satisfies the equality*

$$a_{n+2} = a_{n+1} + \sqrt{a_{n+1} + a_n}$$

for every positive integer n.

(Japan)

Solution 1: The answer is no.

Suppose that there exists a sequence (a_n) of positive integers satisfying the given condition. We will show that this will lead to a contradiction.

For each $n \geq 2$ define $b_n = a_{n+1} - a_n$. Then, by assumption, for $n \geq 2$ we get $b_n = \sqrt{a_n + a_{n-1}}$ so that we have

$$b_{n+1}^2 - b_n^2 = (a_{n+1} + a_n) - (a_n + a_{n-1}) = (a_{n+1} - a_n) + (a_n - a_{n-1}) = b_n + b_{n-1}.$$

Since each a_n is a positive integer we see that b_n is positive integer for $n \geq 2$ and the sequence (b_n) is strictly increasing for $n \geq 3$. Thus $b_n + b_{n-1} = (b_{n+1} - b_n)(b_{n+1} + b_n) \geq b_{n+1} + b_n$, whence $b_{n-1} \geq b_{n+1}$ – a contradiction to increasing of the sequence (b_i).

Thus we conclude that there exists no sequence (a_n) of positive integers satisfying the given condition of the problem.

Solution 2: Suppose that such a sequence exists. We will calculate its members one by one and get a contradiction.

From the equality $a_3 = a_2 + \sqrt{a_2 + a_1}$ it follows that $a_3 > a_2$. Denote positive integers $\sqrt{a_3 + a_2}$ by b and a_3 by a, then we have $\sqrt{2a} > b$. Since $a_4 = a + b$ and $a_5 = a + b + \sqrt{2a + b}$ are positive integers, then $\sqrt{2a + b}$ is positive integer.

Consider $a_6 = a + b + \sqrt{2a + b} + \sqrt{2a + 2b + \sqrt{2a + b}}$. Number $c = \sqrt{2a + 2b + \sqrt{2a + b}}$ must be positive integer, obviously it is greater than $\sqrt{2a + b}$. But

$$(\sqrt{2a + b} + 1)^2 = 2a + b + 2\sqrt{2a + b} + 1 = 2a + 2b + \sqrt{2a + b} + (\sqrt{2a + b} - b) + 1 > c^2.$$

So $\sqrt{2a + b} < c < \sqrt{2a + b} + 1$ which is impossible.

Solutions 3: We will show that there is no sequence (a_n) of positive integers which consists of $N > 5$ members and satisfies

$$a_{n+2} = a_{n+1} + \sqrt{a_{n+1} + a_n} \tag{1}$$

for all $n = 1, \ldots, N - 2$. Moreover, we will describe all such sequences with five members.

Since every a_i is a positive integer it follows from (1) that there exists such positive integer k (obviously k depends on n) that

$$a_{n+1} + a_n = k^2. \tag{2}$$

From (1) we have $(a_{n+2} - a_{n+1})^2 = a_{n+1} + a_n$, consider this equality as a quadratic equation with respect to a_{n+1}:

$$a_{n+1}^2 - (2a_{n+2} + 1)a_{n+1} + a_{n+2}^2 - a_n = 0.$$

Obviously its solutions are $(a_{n+1})_{1,2} = \dfrac{2a_{n+2} + 1 \pm \sqrt{D}}{2}$, where

$$D = 4(a_n + a_{n+2}) + 1. \tag{3}$$

Since $a_{n+2} > a_{n+1}$ we have

$$a_{n+1} = \frac{2a_{n+2} + 1 - \sqrt{D}}{2}.$$

From the last equality, using that a_{n+1} and a_{n+2} are positive integers, we conclude that D is a square of some odd number i.e. $D = (2m+1)^2$ for some positive integer $m \in \mathbb{N}$, substitute this into (3):

$$a_n + a_{n+2} = m(m+1). \tag{4}$$

Now adding a_n to both sides of (1) and using (2) and (4) we get $m(m+1) = k^2 + k$ whence $m = k$. So

$$\begin{cases} a_n + a_{n+1} = k^2, \\ a_n + a_{n+2} = k^2 + k \end{cases} \tag{5}$$

for some positive integer k (recall that k depends on n).

Write equations (5) for $n = 2$ and $n = 3$, then for some positive integers k and ℓ we get

$$\begin{cases} a_2 + a_3 = k^2, \\ a_2 + a_4 = k^2 + k, \\ a_3 + a_4 = \ell^2, \\ a_3 + a_5 = \ell^2 + \ell. \end{cases} \tag{6}$$

Solution of this linear system is

$$a_2 = \frac{2k^2 - \ell^2 + k}{2}, \quad a_3 = \frac{\ell^2 - k}{2}, \quad a_4 = \frac{\ell^2 + k}{2}, \quad a_5 = \frac{\ell^2 + 2\ell + k}{2}. \tag{7}$$

From $a_2 < a_4$ we obtain $k^2 < \ell^2$ hence $k < \ell$.

Consider a_6:

$$a_6 = a_5 + \sqrt{a_5 + a_4} = a_5 + \sqrt{\ell^2 + \ell + k}\,.$$

Since $0 < k < l$ we have $\ell^2 < \ell^2 + \ell + k < (\ell+1)^2$. So a_6 cannot be integer i.e. there is no such sequence with six or more members.

To find all required sequences with five members we must find positive integers a_2, a_3, a_4 and a_5 which satisfy (7) for some positive integers $k < \ell$. Its clear that k and ℓ must be of the same parity. Vise versa, let positive integers k, ℓ be of the same parity and satisfy $k < \ell$ then from (7) we get integers a_2, a_3, a_4 and a_5 then $a_1 = (a_3 - a_2)^2 - a_2$ and it remains to verify that a_1 and a_2 are positive i.e. $2k^2 + k > \ell^2$ and $2(\ell^2 - k^2 - k)^2 > 2k^2 - \ell^2 + k$.

Solution 4: It is easy to see that (a_n) is increasing for large enough n. Hence

$$a_{n+1} < a_n + \sqrt{2a_n} \tag{1}$$

and

$$a_n < a_{n-1} + \sqrt{2a_{n-1}}. \tag{2}$$

Lets define $b_n = a_n + a_{n-1}$. Using AM-QM inequality we have

$$\frac{\sqrt{2a_n} + \sqrt{2a_{n-1}}}{2} \le \sqrt{\frac{2a_n + 2a_{n-1}}{2}}. \tag{3}$$

Adding (1), (2) and using (3):

$$b_{n+1} < b_n + \sqrt{2a_n} + \sqrt{2a_{n-1}} \le b_n + 2\sqrt{b_n}.$$

Let $b_n = m^2$. Since (b_n) is increasing for large enough n, we have:

$$m^2 < b_{n+1} < m^2 + 2m < (m+1)^2.$$

So, b_{n+1} can't be a perfect square, so we get contradiction.

Problem 5. *Let m, n be positive integers with $m > 1$. Anastasia partitions the integers $1, 2, \ldots, 2m$ into m pairs. Boris then chooses one integer from each pair and finds the sum of these chosen integers. Prove that Anastasia can select the pairs so that Boris cannot make his sum equal to n.*

(Netherlands)

Solution 1A: Define the following ordered partitions:

$$P_1 = (\{1,2\}, \{3,4\}, \ldots, \{2m-1, 2m\}),$$
$$P_2 = (\{1, m+1\}, \{2, m+2\}, \ldots, \{m, 2m\}),$$
$$P_3 = (\{1, 2m\}, \{2, m+1\}, \{3, m+2\}, \ldots, \{m, 2m-1\}).$$

For each P_j we will compute the possible values for the expression $s = a_1 + \ldots + a_m$, where $a_i \in P_{j,i}$ are the chosen integers. Here, $P_{j,i}$ denotes the i-th coordinate of the ordered partition P_j.
We will denote by σ the number $\sum_{i=1}^{m} i = (m^2+m)/2$.

- Consider the partition P_1 and a certain choice with corresponding sum s. We find that

  $$m^2 = \sum_{i=1}^{m}(2i-1) \le s \le \sum_{i=1}^{m} 2i = m^2 + m.$$

  Hence, if $n < m^2$ or $n > m^2 + m$, this partition gives a positive answer.

- Consider the partition P_2 and a certain choice with corresponding s. We find that

  $$s \equiv \sum_{i=1}^{m} i \equiv \sigma \pmod{m}.$$

  Hence, if $m^2 \le n \le m^2 + m$ and $n \not\equiv \sigma \pmod{m}$, this partition solves the problem.

- Consider the partition P_3 and a certain choice with corresponding s. We set

  $$d_i = \begin{cases} 0 & \text{if } a_i = i \\ 1, & \text{if } a_i \ne i. \end{cases}$$

  We also put $d = \sum_{i=1}^{m} d_i$, and note that $0 \le d \le m$. Note also that if $a_i \ne i$, then $a_i \equiv i - 1 \pmod{m}$. Hence, for all $a_i \in P_{3,i}$ it holds that

  $$a_i \equiv i - d_i \pmod{m}.$$

  Hence,

  $$s \equiv \sum_{i=1}^{m} a_i \equiv \sum_{i=1}^{m}(i - d_i) \equiv \sigma - d \pmod{m},$$

  which can only be congruent to σ modulo m if all d_i are equal, which forces $s = (m^2+m)/2$ or $s = (3m^2+m)/2$. Since $m > 1$, it holds that

  $$\frac{m^2+m}{2} < m^2 < m^2 + m < \frac{3m^2+m}{2}.$$

  Hence if $m^2 \le n \le m^2 + m$ and $n \equiv \sigma \pmod{m}$, then s cannot be equal to n, so partition P_3 suffices for such n.

Note that all n are treated in one of the cases above, so we are done.

Common notes for solutions 1B and 1C: Given the analysis of P_1 and P_2 as in the solution 1A, we may conclude (noting that $\sigma \equiv m(m+1)/2 \pmod{m}$) that if m is odd then m^2 and m^2+m are the only candidates for counterexamples n, while if m is even then $m^2 + \frac{m}{2}$ is the only candidate.

There are now various ways to proceed as alternatives to the partition P_3.

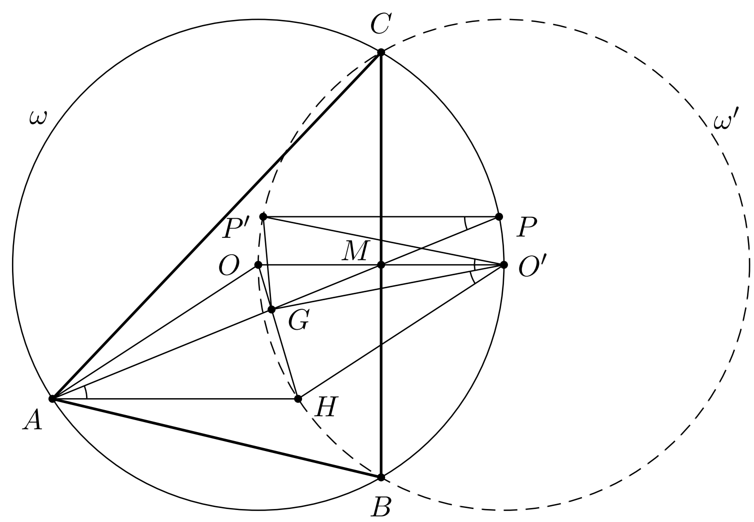
Solution 1B: Consider the partition $(\{1, m+2\}, \{2, m+3\}, \ldots, \{m-1, 2m\}, \{m, m+1\})$. We consider possible sums mod $m+1$. For the first $m-1$ pairs, the elements of each pair are congruent mod $m+1$, so the sum of one element of each pair is (mod $m+1$) congruent to $\frac{1}{2}m(m+1) - m$, which is congruent to 1 if $m+1$ is odd and $1 + \frac{m+1}{2}$ if $m+1$ is even. Now the elements of the last pair are congruent to -1 and 0, so any achievable value of n is congruent to 0 or 1 if $m+1$ is odd, and to 0 or 1 plus $\frac{m+1}{2}$ if $m+1$ is even. If m is even then $m^2 + \frac{m}{2} \equiv 1 + \frac{m}{2}$, which is not congruent to 0 or 1. If m is odd then $m^2 \equiv 1$ and $m^2 + m \equiv 0$, neither of which can equal 0 or 1 plus $\frac{m+1}{2}$.

Solution 1C: Similarly, consider the partition $(\{1, m\}, \{2, m+1\}, \ldots, \{m-1, 2m-2\}, \{2m-1, 2m\})$, this time considering sums of elements of pairs mod $m-1$. If $m-1$ is odd, the sum is congruent to 1 or 2; if $m-1$ is even, to 1 or 2 plus $\frac{m-1}{2}$. If m is even then $m^2 + \frac{m}{2} \equiv 1 + \frac{m}{2}$, and this can only be congruent to 1 or 2 when $m = 2$. If m is odd, m^2 and m^2+m are congruent to 1 and 2, and these can only be congruent to 1 or 2 plus $\frac{m-1}{2}$ when $m = 3$. Now the cases of $m = 2$ and $m = 3$ need considering separately (by finding explicit partitions excluding each n).

Solution 2: This solution does not use modulo arguments. Use only P_1 from the solution 1A to conclude that $m^2 \le n \le m^2 + m$. Now consider the partition $(\{1, 2m\}, \{2, 3\}, \{4, 5\}, \ldots, \{2m-2, 2m-1\})$. If 1 is chosen from the first pair, the sum is at most m^2; if $2m$ is chosen, the sum is at least $m^2 + m$. So either $n = m^2$ or $n = m^2 + m$. Now consider the partition $(\{1, 2m-1\}, \{2, 2m\}, \{3, 4\}, \{5, 6\}, \ldots, \{2m-3, 2m-2\})$. Sums of one element from each of the last $m-2$ pairs are in the range from $(m-2)m = m^2 - 2m$ to $(m-2)(m+1) = m^2 - m - 2$ inclusive. Sums of one element from each of the first two pairs are 3, $2m+1$ and $4m-1$. In the first case we have $n \le m^2 - m + 1 < m^2$, in the second $m^2 + 1 \le n \le m^2 + m - 1$ and in the third $n \ge m^2 + 2m - 1 > m^2 + m$. So these three partitions together have eliminated all n.

Problem 6. *Let H be the orthocenter and G be the centroid of acute-angled triangle $\triangle ABC$ with $AB \neq AC$. The line AG intersects the circumcircle of $\triangle ABC$ at A and P. Let P' be the reflection of P in the line BC. Prove that $\angle CAB = 60^\circ$ if and only if $HG = GP'$.*

(Ukraine)



Solution 1: Let ω be the circumcircle of $\triangle ABC$. Reflecting ω in line BC, we obtain circle ω' which, obviously, contains points H and P'. Let M be the midpoint of BC. As triangle $\triangle ABC$ is acute-angled, then H and O lie inside this triangle.

Let us assume that $\angle CAB = 60^\circ$. Since

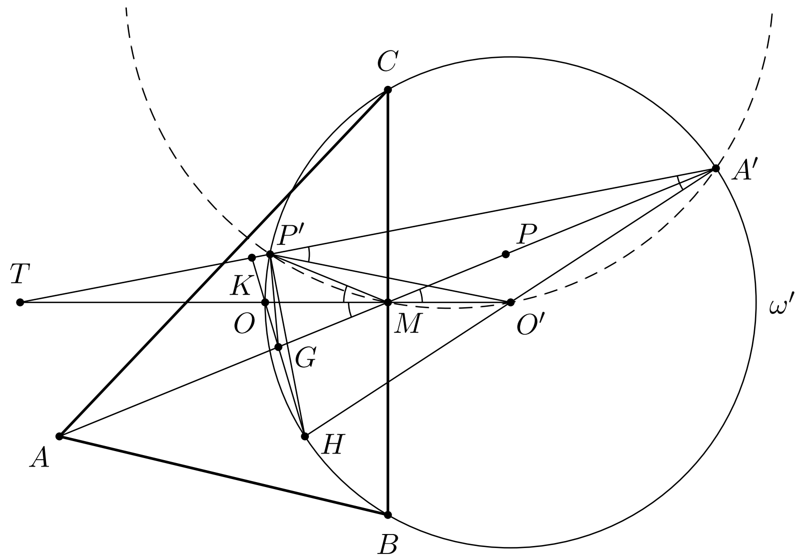
$$\angle COB = 2\angle CAB = 120^\circ = 180^\circ - 60^\circ = 180^\circ - \angle CAB = \angle CHB,$$

hence O lies on ω'. Reflecting O in line BC, we obtain point O' which lies on ω and this point is the center of ω'. Then $OO' = 2OM = 2R\cos\angle CAB = AH$, so $AH = OO' = HO' = AO = R$, where R is the radius of ω and, naturally, of ω'. Then quadrilateral $AHO'O$ is a rhombus, so A and O' are symmetric to each other with respect to HO. As H, G and O are collinear (Euler line), then $\angle GAH = \angle HO'G$. Diagonals of quadrilateral $GOPO'$ intersects at M. Since $\angle BOM = 60^\circ$, so

$$OM = MO' = \operatorname{ctg} 60^\circ \cdot MB = \frac{MB}{\sqrt{3}}.$$

As $3MO \cdot MO' = MB^2 = MB \cdot MC = MP \cdot MA = 3MG \cdot MP$, then $GOPO'$ is a cyclic. Since BC is a perpendicular bisector of OO', so the circumcircle of quadrilateral $GOPO'$ is symmetrical with respect to BC. Thus P' also belongs to the circumcircle of $GOPO'$, hence $\angle GO'P' = \angle GPP'$. Note that $\angle GPP' = \angle GAH$ since $AH || PP'$. And as it was proved $\angle GAH = \angle HO'G$, then $\angle HO'G = \angle GO'P'$. Thus triangles $\triangle HO'G$ and $\triangle GO'P'$ are equal and hence $HG = GP'$.

Now we will prove that if $HG = GP'$ then $\angle CAB = 60^\circ$. Reflecting A with respect to M, we get A'. Then, as it was said in the first part of solution, points B, C, H and P' belong to ω'. Also it is clear that A' belongs to ω'. Note that $HC \perp CA'$ since $AB||CA'$ and hence HA' is a diameter of ω'. Obviously, the center O' of circle ω' is midpoint of HA'. From $HG = GP'$ it follows that $\triangle HGO'$ is equal to $\triangle P'GO'$. Therefore H and P' are symmetric with respect to GO'. Hence $GO' \perp HP'$ and $GO'||A'P'$. Let HG intersect $A'P'$ at K and $K \not\equiv O$ since $AB \neq AC$. We conclude that $HG = GK$, because line GO' is midline of the triangle $\triangle HKA'$. Note that $2GO = HG$. since HO is Euler line of triangle ABC. So O is midpoint of segment GK. Because of $\angle CMP = \angle CMP'$, then $\angle GMO = \angle OMP'$. Line OM, that passes through O', is an external angle bisector of $\angle P'MA'$. Also we know that $P'O' = O'A'$, then O' is the midpoint of arc $P'MA'$ of the circumcircle of triangle $\triangle P'MA'$. It



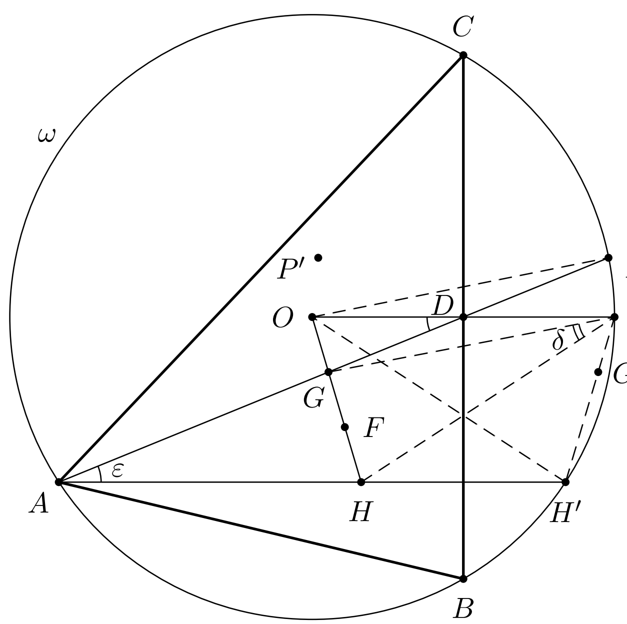
follows that quadrilateral $P'MO'A'$ is cyclic, then $\angle O'MA' = \angle O'P'A' = \angle O'A'P'$. Let OM and $P'A'$ intersect at T. Triangles $\triangle TO'A'$ and $\triangle A'O'M$ are similar, hence $O'A'/O'M = O'T/O'A'$. In the other words, $O'M \cdot O'T = O'A'^2$. Using Menelaus' theorem for triangle $\triangle HKA'$ and line TO', we obtain that

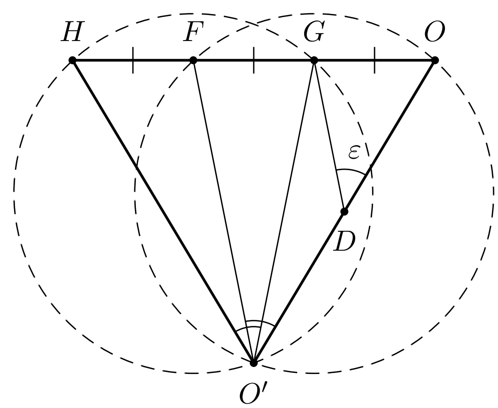
$$\frac{A'O'}{O'H} \cdot \frac{HO}{OK} \cdot \frac{KT}{TA'} = 3 \cdot \frac{KT}{TA'} = 1.$$

It follows that $KT/TA' = 1/3$ and $KA' = 2KT$. Using Menelaus' theorem for triangle $TO'A'$ and line HK we get that

$$1 = \frac{O'H}{HA'} \cdot \frac{A'K}{KT} \cdot \frac{TO}{OO'} = \frac{1}{2} \cdot 2 \cdot \frac{TO}{OO'} = \frac{TO}{OO'}.$$

It means that $TO = OO'$, so $O'A'^2 = O'M \cdot O'T = OO'^2$. Hence $O'A' = OO'$ and, consequently, $O \in \omega'$. Finally we conclude that $2\angle CAB = \angle BOC = 180° - \angle CAB$, so $\angle CAB = 60°$.





Solution 2: Let O' and G' denote the reflection of O and G, respectively, with respect to the line BC. We then need to show $\angle CAB = 60°$ iff $G'H' = G'P$. Note that $\triangle H'OP$ is isosceles and hence

$G'H' = G'P$ is equivalent to G' lying on the bisector $\angle H'OP$. Let $\angle H'AP = \varepsilon$. By the assumption $AB \neq AC$, we have $\varepsilon \neq 0$. Then $\angle H'OP = 2\angle H'AP = 2\varepsilon$, hence $G'H' = G'P$ iff $\angle G'OH' = \varepsilon$. But $\angle GO'H = \angle G'OH'$. Let D be the midpoint of OO'. It is known that $\angle GDO = \angle GAH = \varepsilon$. Let F be the midpoint of HG. Then $HG = FO$ (Euler line). Let $\angle GO'H = \delta$. We then have to show $\delta = \varepsilon$ iff $\angle CAB = 60°$. But by similarity ($\triangle GDO \sim \triangle FO'O$) we have $\angle FO'O = \varepsilon$. Consider the circumcircles of the triangles $FO'O$ and $GO'H$. By the sine law and since the segments HG and FO are of equal length we deduce that the circumcircles of the triangles $FO'O$ and $GO'H$ are symmetric with respect to the perpendicular bisector of the segment FG iff $\delta = \varepsilon$. Obviously, O' is the common point of these two circles. Hence O' must be fixed after the symmetry about the perpendicular bisector of the segment FG iff $\delta = \varepsilon$ so we have $\varepsilon = \delta$ iff $\triangle HOO'$ is isosceles. But $HO' = H'O = R$, and so

$$\varepsilon = \delta \iff OO' = R \iff OD = \frac{R}{2} \iff \cos\angle CAB = \frac{1}{2} \iff \angle CAB = 60°.$$

Solution 3: Let H' and G' denote the reflection of points H and G with respect to the line BC. It is known that H' belongs to the circumcircle of $\triangle ABC$. The equality $HG = GP'$ is equivalent to $H'G' = G'P$. As in the **Solution 2**, it is equivalent to the statement that point G' belongs to the perpendicular bisector of $H'P$, which is equivalent to $OG' \perp H'P$, where O is the circumcenter of $\triangle ABC$.

Let points $A(a)$, $B(b)$, and $C(c = -\overline{b})$ belong to the unit circle in the complex plane. Point G have coordinate $g = (a + b - \overline{b})/3$. Since BC is parallel to the real axis point H' have coordinate $h' = \overline{a} = 1/a$.

Point $P(p)$ belongs to the unit circle, so $\overline{p} = 1/p$. Since a, p, g are collinear we have $\frac{p-a}{g-a} = \overline{\left(\frac{p-a}{g-a}\right)}$. After computation we get $p = \frac{g-a}{1-\overline{g}a}$. Since $G'(g)$ is the reflection of G with respect to the chord BC, we have $g' = b + (-\overline{b}) - b(-\overline{b})\overline{g} = b - \overline{b} + \overline{g}$. Let $b - \overline{b} = d$. We have $\overline{d} = -d$. So

$$g = \frac{a+d}{3}, \quad \overline{g} = \frac{\overline{a}-d}{3}, \quad g' = d + \overline{g} = \frac{\overline{a}+2d}{3}, \quad \overline{g'} = \frac{a-2d}{3} \quad \text{and} \quad p = \frac{g-a}{1-\overline{g}a} = \frac{d-2a}{2+ad}. \tag{1}$$

It is easy to see that $OG' \perp H'P'$ is equivalent to

$$\frac{g'}{h'-p} = -\overline{\left(\frac{g'}{h'-p}\right)} = -\frac{g'}{\frac{1}{h'} - \frac{1}{p}} = \frac{\overline{g'}h'p}{h'-p}$$

since h' and p belong to the unit circle (note that $H' \neq P$ because $AB \neq AC$). This is equivalent to $g' = \overline{g'}h'p$ and from (1), after easy computations, this is equivalent to $a^2g^2 + a^2 + d^2 + 1 = (a^2+1)(d^2+1) = 0$.

We cannot have $a^2 + 1 = 0$, because then $a = \pm i$, but $AB \neq AC$. Hence $d = b - \overline{b} = \pm i$, and the pair $\{b, c = -\overline{b}\}$ is either $\{-\sqrt{3}/2 + i/2, \sqrt{3}/2 + i/2\}$ or $\{-\sqrt{3}/2 - i/2, \sqrt{3}/2 - i/2\}$. Both cases are equivalent to $\angle BAC = 60°$ which completes the proof.