Thursday, April 16, 2015

Problem 1. Let $\triangle ABC$ be an acute-angled triangle, and let D be the foot of the altitude from C. The angle bisector of $\angle ABC$ intersects CD at E and meets the circumcircle ω of triangle $\triangle ADE$ again at F. If $\angle ADF = 45°$, show that CF is tangent to ω.

Problem 2. A *domino* is a 2×1 or 1×2 tile. Determine in how many ways exactly n^2 dominoes can be placed without overlapping on a $2n \times 2n$ chessboard so that every 2×2 square contains at least two uncovered unit squares which lie in the same row or column.

Problem 3. Let n, m be integers greater than 1, and let $a_1, a_2, \ldots, a_m$ be positive integers not greater than n^m. Prove that there exist positive integers $b_1, b_2, \ldots, b_m$ not greater than n, such that

$$\gcd(a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) < n,$$

where $\gcd(x_1, x_2, \ldots, x_m)$ denotes the greatest common divisor of $x_1, x_2, \ldots, x_m$.

Time: 4 hours and 30 minutes
Each problem is worth 7 points