Language: **English**

Day: **1**

Saturday, April 11, 2026

Problem 1. A 2026×2026 board is said to be *bordeaux* if at least one of its 2026^2 unit cells is coloured red. A rectangular region made of cells is *oddly-rectangular* if it contains an odd number of red cells. Determine the largest positive integer M such that, in every possible 2026×2026 bordeaux board, there exists an oddly-rectangular region of at least M cells.

Note: A rectangular region has sides that are parallel to the sides of the board and contains all of its interior.

Problem 2. Given a positive integer n, Marie plays a game where she starts with the number 1 on a blackboard. As many times as she wants, she can choose an integer j such that $1 \leq j \leq n$ and replace the number V on the blackboard with the number $j \cdot R\left(\frac{V}{j}\right)$. Here, $R(x)$ denotes the nearest integer to x; if x is exactly halfway between two consecutive integers, it is rounded up. For example, $R(1.3) = 1$ and $R(1.5) = R(1.8) = 2$.

a) Prove that for each given n, there is a positive integer B such that Marie can never end up with a number larger than B on the blackboard.

b) For any given n, let $f(n)$ be the maximum number obtainable on the blackboard after finitely many replacements. Show that there exists a positive integer N such that for all $n \geq N$, we have that 2026 divides $f(n)$.

Problem 3. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y, the following holds:

$$f((f(x) + f(y))^2) = (x + y)f(x + y).$$

Time: 4 hours and 30 minutes
Each problem is worth 7 points