Problem 4. For a sequence $a_1 < a_2 < \cdots < a_n$ of integers, a pair $(a_i, a_j)$ with $1 \leq i < j \leq n$ is called interesting if there exists a pair $(a_k, a_\ell)$ of integers with $1 \leq k < \ell \leq n$ such that

$$\frac{a_\ell - a_k}{a_j - a_i} = 2.$$ 

For each $n \geq 3$, find the largest possible number of interesting pairs in a sequence of length $n$.

Problem 5. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f: \mathbb{N} \to \mathbb{N}$ such that the following conditions are true for every pair of positive integers $(x, y)$:

(i) $x$ and $f(x)$ have the same number of positive divisors.

(ii) If $x$ does not divide $y$ and $y$ does not divide $x$, then

$$\gcd(f(x), f(y)) > f(\gcd(x, y)).$$

Here $\gcd(m, n)$ is the largest positive integer that divides both $m$ and $n$.

Problem 6. Find all positive integers $d$ for which there exists a degree $d$ polynomial $P$ with real coefficients such that there are at most $d$ different values among $P(0), P(1), P(2), \ldots, P(d^2 - d)$. 

Language: English  
Time: 4 hours and 30 minutes  
Each problem is worth 7 points