Saturday, April 13, 2024

Problem 1. Two different integers u and v are written on a board. We perform a sequence of steps. At each step we do one of the following two operations:

(i) If a and b are different integers on the board, then we can write $a+b$ on the board, if it is not already there.

(ii) If a, b and c are three different integers on the board, and if an integer x satisfies $ax^2+bx+c=0$, then we can write x on the board, if it is not already there.

Determine all pairs of starting numbers (u, v) from which any integer can eventually be written on the board after a finite sequence of steps.

Problem 2. Let ABC be a triangle with $AC > AB$, and denote its circumcircle by Ω and incentre by I. Let its incircle meet sides BC, CA, AB at D, E, F respectively. Let X and Y be two points on minor arcs $\widehat{DF}$ and $\widehat{DE}$ of the incircle, respectively, such that $\angle BXD = \angle DYC$. Let line XY meet line BC at K. Let T be the point on Ω such that KT is tangent to Ω and T is on the same side of line BC as A. Prove that lines TD and AI meet on Ω.

Problem 3. We call a positive integer n *peculiar* if, for any positive divisor d of n, the integer $d(d+1)$ divides $n(n+1)$. Prove that for any four different peculiar positive integers A, B, C and D, the following holds:

$$\gcd(A, B, C, D) = 1.$$

Here $\gcd(A, B, C, D)$ *is the largest positive integer that divides all of* A, B, C *and* D.

Time: 4 hours and 30 minutes
Each problem is worth 7 points