EGMO 2023 – Problems and solutions

Problems

Problem 1. There are \( n \geq 3 \) positive real numbers \( a_1, a_2, \ldots, a_n \). For each \( 1 \leq i \leq n \) we let \( b_i = \frac{a_{i-1} + a_{i+1}}{a_i} \) (here we define \( a_0 \) to be \( a_n \) and \( a_{n+1} \) to be \( a_1 \)). Assume that for all \( i \) and \( j \) in the range \( 1 \) to \( n \), we have \( a_i \leq a_j \) if and only if \( b_i \leq b_j \).

Prove that \( a_1 = a_2 = \cdots = a_n \).

Problem 2. We are given an acute triangle \( ABC \). Let \( D \) be the point on its circumcircle such that \( AD \) is a diameter. Suppose that points \( K \) and \( L \) lie on segments \( AB \) and \( AC \), respectively, and that \( DK \) and \( DL \) are tangent to circle \( AKL \).

Show that line \( KL \) passes through the orthocentre of \( ABC \).

The altitudes of a triangle meet at its orthocentre.

Problem 3. Let \( k \) be a positive integer. Lexi has a dictionary \( D \) consisting of some \( k \)-letter strings containing only the letters \( A \) and \( B \). Lexi would like to write either the letter \( A \) or the letter \( B \) in each cell of a \( k \times k \) grid so that each column contains a string from \( D \) when read from top-to-bottom and each row contains a string from \( D \) when read from left-to-right.

What is the smallest integer \( m \) such that if \( D \) contains at least \( m \) different strings, then Lexi can fill her grid in this manner, no matter what strings are in \( D \)?

Problem 4. Turbo the snail sits on a point on a circle with circumference 1. Given an infinite sequence of positive real numbers \( c_1, c_2, c_3, \ldots \), Turbo successively crawls distances \( c_1, c_2, c_3, \ldots \) around the circle, each time choosing to crawl either clockwise or counterclockwise.

For example, if the sequence \( 0.4, 0.6, 0.3, \ldots \), then Turbo may start crawling as follows:

![Diagram of Turbo crawling around a circle](image)

Determine the largest constant \( C > 0 \) with the following property: for every sequence of positive real numbers \( c_1, c_2, c_3, \ldots \) with \( c_i < C \) for all \( i \), Turbo can (after studying the sequence) ensure that there is some point on the circle that it will never visit or crawl across.

Problem 5. We are given a positive integer \( s \geq 2 \). For each positive integer \( k \), we define its twist \( k' \) as follows: write \( k \) as \( as + b \), where \( a, b \) are non-negative integers and \( b < s \), then \( k' = bs + a \). For the positive integer \( n \), consider the infinite sequence \( d_1, d_2, \ldots \) where \( d_1 = n \) and \( d_{i+1} \) is the twist of \( d_i \) for each positive integer \( i \).
Prove that this sequence contains 1 if and only if the remainder when $n$ is divided by $s^2 - 1$ is either 1 or $s$.

**Problem 6.** Let $ABC$ be a triangle with circumcircle $\Omega$. Let $S_b$ and $S_c$ respectively denote the midpoints of the arcs $AC$ and $AB$ that do not contain the third vertex. Let $N_a$ denote the midpoint of arc $BAC$ (the arc $BC$ containing $A$). Let $I$ be the incentre of $ABC$. Let $\omega_b$ be the circle that is tangent to $AB$ and internally tangent to $\Omega$ at $S_b$, and let $\omega_c$ be the circle that is tangent to $AC$ and internally tangent to $\Omega$ at $S_c$. Show that the line $IN_a$, and the line through the intersections of $\omega_b$ and $\omega_c$, meet on $\Omega$.

*The incentre of a triangle is the centre of its incircle, the circle inside the triangle that is tangent to all three sides.*
Solutions

**Problem 1.** There are $n \geq 3$ positive real numbers $a_1, a_2, \ldots, a_n$. For each $1 \leq i \leq n$ we let $b_i = \frac{a_{i-1} + a_{i+1}}{a_i}$ (here we define $a_0$ to be $a_n$ and $a_{n+1}$ to be $a_1$). Assume that for all $i$ and $j$ in the range 1 to $n$, we have $a_i \leq a_j$ if and only if $b_i \leq b_j$.

Prove that $a_1 = a_2 = \cdots = a_n$.

**Solution 1.** Suppose that not all $a_i$ are equal. Consider an index $i$ such that $a_i$ is maximal and $a_{i+1} < a_i$. Then

$$b_i = \frac{a_{i-1} + a_{i+1}}{a_i} < \frac{2a_i}{a_i} = 2.$$ 

But since $a_i$ is maximal, $b_i$ is also maximal, so we must have $b_j < 2$ for all $j \in \{1, 2, \ldots, n\}$. However, consider the product $b_1b_2 \cdots b_n$. We have

$$b_1b_2 \cdots b_n = \frac{a_n + a_2}{a_1} \cdot \frac{a_1 + a_3}{a_2} \cdot \cdots \cdot \frac{a_{n-1} + a_1}{a_n} 
\geq 2^n \sqrt[n]{\frac{a_n a_2}{a_1} \cdot \frac{a_1 a_3}{a_2} \cdots \frac{a_{n-1} a_1}{a_n}} 
= 2^n,$$ 

where we used the inequality $x + y \geq 2\sqrt{xy}$ for $x = a_{i-1}$, $y = a_{i+1}$ for all $i \in \{1, 2, \ldots, n\}$ in the second row.

Since the product of all $b_i$ is at least $2^n$, at least one of them must be greater than 2, which is a contradiction with the previous conclusion.

Thus, all $a_i$ must be equal. \qed

**Solution 2.** This is a version of Solution 1 without use of proof by contradiction.

Taking $a_i$ such that it is maximal among $a_1, \ldots, a_n$, we obtain $b_i \leq 2$. Thus $b_j \leq 2$ for all $j \in \{1, 2, \ldots, n\}$.

The second part of Solution 1 then gives $2^n \geq b_1 \cdots b_n \geq 2^n$, which together with $b_j \leq 2$ for all $j \in \{1, 2, \ldots, n\}$ implies that $b_j = 2$ for all $j \in \{1, 2, \ldots, n\}$. Since we have $b_1 = b_2 = \cdots = b_n$, the condition that $a_i \leq a_j \iff b_i \leq b_j$ gives that $a_1 = a_2 = \cdots = a_n$. \qed

**Solution 3.** We first show that $b_j \leq 2$ for all $j$ as in Solution 2. Then

$$2n \geq b_1 + \cdots + b_n = \frac{a_n}{a_1} + \frac{a_2}{a_1} + \frac{a_1}{a_2} + \frac{a_3}{a_2} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_1}{a_n} 
\geq 2n \sqrt[n]{\frac{a_n}{a_1} \cdot \frac{a_2}{a_1} \cdot \frac{a_1}{a_2} \cdots \frac{a_{n-1}}{a_n} \cdot \frac{a_1}{a_n}} = 2n \cdot 1 = 2n,$$ 

where we used the AM-GM inequality.

It follows that all $b_j$’s are equal which as in Solution 2 gives $a_1 = a_2 = \cdots = a_n$. \qed
Solution 4. By assumption \(a_i b_i = a_{i-1} + a_{i+1}\) for \(i = \{1, 2, \ldots, n\}\), hence,
\[
\sum_{i=1}^{n} a_i b_i = 2 \sum_{i=1}^{n} a_i.
\]
Since \(a_i \leq a_j\) if and only if \(b_i \leq b_j\), the Chebyshev’s inequality implies
\[
\left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{i=1}^{n} b_i\right) \leq n \cdot \sum_{i=1}^{n} a_i b_i = 2n \cdot \sum_{i=1}^{n} a_i.
\]
and so \(\sum_{i=1}^{n} b_i \leq 2n\). On the other hand, we have
\[
\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} \frac{a_{i-1}}{a_i} + \sum_{i=1}^{n} \frac{a_{i+1}}{a_i} = \sum_{i=1}^{n} \frac{a_{i-1}}{a_i} + \sum_{i=1}^{n} \frac{a_i}{a_{i-1}} = \sum_{i=1}^{n} \left(\frac{a_{i-1}}{a_i} + \frac{a_i}{a_{i-1}}\right),
\]
so we can use the AM-GM inequality to estimate
\[
\sum_{i=1}^{n} b_i \geq \sum_{i=1}^{n} 2 \sqrt{\frac{a_{i-1}}{a_i} \cdot \frac{a_i}{a_{i-1}}} = 2n.
\]
We conclude that we must have equalities in all the above, which implies \(\frac{a_{i-1}}{a_i} = \frac{a_i}{a_{i-1}}\) and consequently \(a_i = a_{i-1}\) for all positive integers \(i\). Hence, all \(a\)'s are equal. \(\square\)

Solution 5. As in Solution 4 we show that \(\sum_{i=1}^{n} b_i \leq 2n\) and as in Solution 1 we show that \(\prod_{i=1}^{n} b_i \geq 2^n\). We now use the AM-GM inequality and the first inequality to get
\[
\prod_{i=1}^{n} b_i \leq \left(\frac{1}{n} \sum_{i=1}^{n} b_i\right)^n \leq \left(\frac{1}{n} \cdot 2n\right)^n = 2^n.
\]
This implies that we must have equalities in all the above. In particular, we have equality in the AM-GM inequality, so all \(b\)'s are equal and as in Solution 2 then all \(a\)'s are equal. \(\square\)

Solution 6. Let \(a_i\) to be minimal and \(a_j\) maximal among all \(a\)'s. Then
\[
b_j = \frac{a_{j-1} + a_{j+1}}{a_i} \leq \frac{2a_j}{a_j} = 2 = \frac{2a_i}{a_i} \leq \frac{a_{i-1} + a_{i+1}}{a_i} = b_i
\]
and by assumption \(b_i \leq b_j\). Hence, we have equalities in the above so \(b_j = 2\) so \(a_{j-1} + a_{j+1} = 2a_j\) and therefore \(a_{j-1} = a_j = a_{j+1}\). We have thus shown that the two neighbors of a maximal \(a\) are also maximal. By an inductive argument all \(a\)'s are maximal, hence equal. \(\square\)

Solution 7. Choose an arbitrary index \(i\) and assume without loss of generality that \(a_i \leq a_{i+1}\). (If the opposite inequality holds, reverse all the inequalities below.) By induction we will show that for each \(k \in \mathbb{N}_0\) the following two inequalities hold
\[
a_{i+1+k} \geq a_{i-k} \quad (1)
\]
\[
a_{i+1+k} a_{i+1-k} \geq a_{i-k} a_{i+k} \quad (2)
\]
(where all indices are cyclic modulo \( n \)). Both inequalities trivially hold for \( k = 0 \).

Assume now that both inequalities hold for some \( k \geq 0 \). The inequality \( a_{i+1+k} \geq a_{i-k} \) implies \( b_{i+1+k} \geq b_{i-k} \), so

\[
\frac{a_{i+k} + a_{i+2+k}}{a_{i+1+k}} \geq \frac{a_{i-1-k} + a_{i+1-k}}{a_{i-k}}.
\]

We may rearrange this inequality by making \( a_{i+2+k} \) the subject so

\[
a_{i+2+k} \geq \frac{a_{i+1+k}a_{i-1-k}}{a_{i-k}} + \frac{a_{i+1-k}a_{i+1+k} - a_{i+k}a_{i-1-k}}{a_{i-k}} \geq \frac{a_{i+1+k}a_{i-1-k}}{a_{i-k}},
\]

where the last inequality holds by (2). It follows that

\[
a_{(i+1)+(k+1)a_{i+1} - (k+1)} \geq a_{i+(k+1)a_{i-1} - (k+1)},
\]

i.e. the inequality (2) holds also for \( k + 1 \). Using (1) we now get

\[
a_{(i+1)+(k+1)} \geq \frac{a_{i+k+1}}{a_{i-k}} a_{i-(k+1)} \geq a_{i-(k+1)},
\]

i.e. (1) holds for \( k + 1 \).

Now we use the inequality (1) for \( k = n - 1 \). We get \( a_i \geq a_{i+1} \), and since at the beginning we assumed \( a_i \leq a_{i+1} \), we get that any two consecutive \( a \)’s are equal, so all of them are equal. \( \square \)

**Solution 8.** We first prove the following claim by induction:

**Claim 1:** If \( a_k a_{k+2} < a_{k+1}^2 \) and \( a_k < a_{k+1} \), then \( a_j a_{j+2} < a_{j+1}^2 \) and \( a_j < a_{j+1} \) for all \( j \).

We assume that \( a_i a_{i+2} < a_{i+1}^2 \) and \( a_i < a_{i+1} \), and then show that \( a_{i-1} a_{i+1} < a_{i}^2 \) and \( a_{i-1} < a_{i} \).

Since \( a_i \leq a_{i+1} \) we have that \( b_i \leq b_{i+1} \). By plugging in the definition of \( b_i \) and \( b_{i+1} \) we have that

\[
a_{i+1} a_{i-1} + a_{i+1}^2 \leq a_i^2 + a_{i+2} a_i.
\]

Using \( a_i a_{i+2} < a_{i+1}^2 \) we get that

\[
a_{i+1} a_{i-1} < a_i^2.
\]

Since \( a_i < a_{i+1} \) we have that \( a_{i-1} < a_i \), which concludes the induction step and hence proves the claim.

We cannot have that \( a_j < a_{j+1} \) for all indices \( j \). Similar as in the above claim, one can prove that if \( a_k a_{k+2} < a_{k+1}^2 \) and \( a_{k+2} < a_{k+1} \), then \( a_{j+1} < a_j \) for all \( j \), which also cannot be the case. Thus we have that \( a_k a_{k+2} \geq a_{k+1}^2 \) for all indices \( k \).

Next observe (e.g. by taking the product over all indices) that this implies \( a_k a_{k+2} = a_{k+1}^2 \) for all indices \( k \), which is equivalent to \( b_k = b_{k+1} \) for all \( k \) and hence \( a_{k+1} = a_k \) for all \( k \). \( \square \)
Solution 9. Define \(c_i := \frac{a_i}{a_{i+1}}\), then \(b_i = c_{i-1} + 1/c_i\). Assume that not all \(c_i\) are equal to 1. Since, \(\prod_{i=1}^n c_i = 1\) there exists a \(k\) such that \(c_k \geq 1\). From the condition given in the problem statement for \((i, j) = (k, k + 1)\) we have
\[
  c_k \geq 1 \iff c_{k-1} + \frac{1}{c_k} \geq c_k + \frac{1}{c_{k+1}} \iff c_{k-1}c_k c_{k+1} + c_{k+1} \geq c_k^2 c_{k+1} + c_k. \tag{5}
\]
Now since \(c_{k+1} \leq c_k^2 c_{k+1}\), it follows that
\[
  c_{i-1}c_{i+1} \geq c_i \implies (c_{i-1} \geq 1 \text{ or } c_{i+1} \geq 1). \tag{6}
\]
So there exist a set of at least 2 consecutive integers, such that the corresponding \(c_i\) are greater or equal to one. By the initial assumption there must exist an index \(\ell\), such that \(c_{\ell-1}, c_\ell \geq 1\) and \(c_{\ell+1} < 1\). We distinguish two cases:

**Case 1:** \(c_\ell > c_{\ell-1} \geq 1\)
From \(c_{\ell-1}c_\ell c_{\ell+1} < c_\ell^2 c_{\ell+1}\) and the inequality (5), we get that \(c_{\ell+1} > c_\ell \geq 1\), which is a contradiction to our choice of \(\ell\).

**Case 2:** \(c_{\ell-1} \geq c_\ell \geq 1\)
Once again looking at the inequality (5) we can find that
\[
  c_{\ell-2} c_{\ell-1} c_\ell \geq c_{\ell-1}^2 c_\ell \implies c_{\ell-2} \geq c_{\ell-1}. \tag{7}
\]
Note that we only needed \(c_{\ell-1} \geq c_\ell \geq 1\) to show \(c_{\ell-2} \geq c_{\ell-1} \geq 1\). So using induction we can easily show \(c_{\ell-s+1} \geq c_{\ell-s}\) for all \(s\).

So
\[
  c_1 \leq c_2 \leq \cdots \leq c_n \leq c_1 \tag{8}
\]
a contradiction to our initial assumption.

So our initial assumption must have been wrong, which implies that all the \(a_i\) must have been equal from the start. \(\square\)
**Problem 2.** We are given an acute triangle $ABC$. Let $D$ be the point on its circumcircle such that $AD$ is a diameter. Suppose that points $K$ and $L$ lie on segments $AB$ and $AC$, respectively, and that $DK$ and $DL$ are tangent to circle $AKL$.

Show that line $KL$ passes through the orthocentre of $ABC$.

The altitudes of a triangle meet at its orthocentre.

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**Solution 1.** Let $M$ be the midpoint of $KL$. We will prove that $M$ is the orthocentre of $ABC$. Since $DK$ and $DL$ are tangent to the same circle, $|DK| = |DL|$ and hence $DM \perp KL$. The theorem of Thales in circle $ABC$ also gives $DB \perp BA$ and $DC \perp CA$. The right angles then give that quadrilaterals $BDMK$ and $DMLC$ are cyclic.

If $\angle BAC = \alpha$, then clearly $\angle DKM = \angle LMD = \alpha$ by angle in the alternate segment of circle $AKL$, and so $\angle MDK = \angle LDM = \frac{\pi}{2} - \alpha$, which thanks to cyclic quadrilaterals gives $\angle MBK = \angle LCM = \frac{\pi}{2} - \alpha$. From this, we have $BM \perp AC$ and $CM \perp AB$, and so $M$ indeed is the orthocentre of $ABC$.

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**Solution 2.** Preliminaries

Let $ABC$ be a triangle with circumcircle $\Gamma$. Let $X$ be a point in the plane. The Simson line (Wallace-Simson line) is defined via the following theorem. Drop perpendiculars from $X$ to each of the three side lines of $ABC$. The feet of these perpendiculars are collinear (on
the Simson line of $X$) if and only if $X$ lies on $\Gamma$. The Simson line of $X$ in the circumcircle bisects the line segment $XH$ where $H$ is the orthocentre of triangle $ABC$. See Figure 2.

![Figure 2: The Wallace-Simson configuration](image)

When $X$ is on $\Gamma$, we can enlarge from $X$ with scale factor 2 (a homothety) to take the Simson line to the doubled Simson line which passes through the orthocentre $H$ and contains the reflections of $X$ in each of the three sides of $ABC$.

**Solution of the problem**

![Figure 3: Three circles do the work](image)

Let $\Gamma$ be the circle $ABC$, $\Sigma$ be the circle $AKL$ with centre $O$, and $\Omega$ be the circle on diameter $OD$ so $K$ and $L$ are on this circle by converse of Thales. Let $\Omega$ and $\Gamma$ meet at $D$ and $F$. By Thales in both circles, $\angle AFD$ and $\angle OFD$ are both right angles so $AOF$ is a line. Let $AF$ meet $\Sigma$ again at $T$ so $AT$ (containing $O$) is a diameter of this circle and by Thales, $TL \perp AC$.

Let $G$ (on $\Sigma$) be the reflection of $K$ in $AF$. Now $AT$ is the internal angle bisector of $\angle GAK$ so, by an upmarket use of angles in the same segment (of $\Sigma$), $TL$ is the internal
angle bisector of $\angle GLK$. Thus the line $GL$ is the reflection of the line $KL$ in $TL$, and so also the reflection of $KL$ in the line $AC$ (internal and external angle bisectors).

Our next project is to show that $LGF$ are collinear. Well $\angle FLK = \angle FOK$ (angles in the same segment of $\Omega$) and $\angle GLK = \angle GAK$ (angles in the same segment of $\Sigma$) = $2\angle OAK$ ($AKG$ is isosceles with apex $A$) = $\angle TOK$ (since $OAK$ is isosceles with apex $O$, and this is an external angle at $O$). The point $T$ lies in the interior of the line segment $FO$ so $\angle TOK = \angle FOK$. Therefore $\angle FLK = \angle GLK$ so $LGF$ is a line.

Now from the second paragraph, $F$ is on the reflection of $KL$ in $AC$. By symmetry, $F$ is also on the reflection of $KL$ in $AB$. Therefore the reflections of $F$ in $AB$ and $AC$ are both on $KL$ which must therefore be the doubled Wallace-Simson line of $F$. Therefore the orthocentre of $ABC$ lies on $KL$.

\[\square\]

**Solution 3.** Let $H$ be the orthocentre of triangle $ABC$ and $\Sigma$ the circumcircle of $AKL$ with centre $O$. Let $\Omega$ be the circle with diameter $OD$, which contains $K$ and $L$ by Thales, and let $\Gamma$ be the circumcircle of $ABC$ containing $D$. Denote the second intersection of $\Omega$ and $\Gamma$ by $F$. Since $OD$ and $AD$ are diameters of $\Omega$ and $\Gamma$ we have $\angle OFD = \frac{\pi}{2} = \angle AFD$, so the points $A, O, F$ are collinear. Let $M$ and $N$ be the second intersections of $CH$ and $BH$ with $\Gamma$, respectively. It is well-known that $M$ and $N$ are the reflections of $H$ in $AB$ and $AC$, respectively (because $\angle NCA = \angle NBA = \angle ACM = \angle ABM$). By collinearity of $A, O, F$ and the angles in $\Gamma$ we have

$$\angle NFO = \angle NFA = \angle NBA = \frac{\pi}{2} - \angle BAC = \frac{\pi}{2} - \angle KAL.$$  

Since $DL$ is tangent to $\Sigma$ we obtain

$$\angle NFO = \frac{\pi}{2} - \angle KLD = \angle LDO,$$

where the last equality follows from the fact that $OD$ is bisector of $\angle LDK$ since $LD$
and $KD$ are tangent to $\Sigma$. Furthermore, $\angle LDO = \angle LFO$ since these are angles in $\Omega$. Hence, $\angle NFO = \angle LFO$, which implies that points $N, L, F$ are collinear. Similarly points $M, K, F$ are collinear. Since $N$ and $M$ are reflections of $H$ in $AC$ and $AB$ we have

$$\angle LHN = \angle HNL = \angle BNF = \angle BMF = \angle BMK = \angle KHB.$$  

Hence,  

$$\angle LHK = \angle LHN + \angle NHK = \angle KHB + \angle NHK = \pi$$

and the points $L, H, K$ are collinear.  

Solution 4. As in Solution 3 let $M$ and $N$ be the reflections of the orthocentre in $AB$ and $AC$. Let $\angle BAC = \alpha$. Then $\angle NDM = \pi - \angle MAN = \pi - 2\alpha$.

Let $MK$ and $NL$ intersect at $F$. See Figure 3.

Claim. $\angle NFM = \pi - 2\alpha$, so $F$ lies on the circumcircle.

Proof. Since $KD$ and $LD$ are tangents to circle $AKL$, we have $|DK| = |DL|$ and $\angle DKL = \angle KLD = \alpha$, so $\angle LDK = \pi - 2\alpha$.

By definition of $M$, $N$ and $D$, $\angle MND = \angle AND - \angle ANM = \frac{\pi}{2} - (\frac{\pi}{2} - \alpha) = \alpha$ and analogously $\angle DMN = \alpha$. Hence $|DM| = |DN|$.

From $\angle NDM = \angle LDK = \pi - 2\alpha$ if follows that $\angle LDN = \angle KDM$. Since $|DK| = |DL|$ and $|DM| = |DN|$, triangles $MDK$ and $NDL$ are related by a rotation about $D$ through angle $\pi - 2\alpha$, and hence the angle between $MK$ and $NL$ is $\pi - 2\alpha$, which proved the claim.  

We now finish as in Solution 3:

$$\angle MHK = \angle KMH = \angle FMC = \angle FAC,$$

$$\angle LHN = \angle HNL = \angle BNF = \angle BAF.$$  

As $\angle BAF + \angle FAC = \alpha$, we have $\angle LHK = \alpha + \angle NHM = \alpha + \pi - \alpha = \pi$, so $H$ lies on $KL$.  

Solution 5. Since $AD$ is a diameter, it is well known that $DBHC$ is a parallelogram (indeed, both $BD$ and $CH$ are perpendicular to $AB$, hence parallel, and similarly for $DC \parallel BH$). Let $B', C'$ be the reflections of $D$ in lines $AKB$ and $ALC$, respectively; since $ABD$ and $ACD$ are right angles, these are also the factor-2 homotheties of $B$ and $C$ with respect to $D$, hence $H$ is the midpoint of $B'C'$. We will prove that $B'KC'L$ is a parallelogram: it will then follow that the midpoint of $B'C'$, which is $H$, is also the midpoint of $KL$, and in particular is on the line, as we wanted to show.

We will prove $B'KC'L$ is a parallelogram by showing that $B'K$ and $C'L$ are the same length and direction. Indeed, for lengths we have $KB' = KD = LD = LC'$, where the first and last equalities arise from the reflections defining $B'$ and $C'$, and the middle one
is equality of tangents. For directions, let $\alpha, \beta, \gamma$ denote the angles of triangle $AKL$. Immediate angle chasing in the circle $AKL$, and the properties of the reflections, yield

$$\angle C'L'C = \angle CLD = \angle AKL = \beta$$
$$\angle BK'B' = \angle DKB = \angle KLA = \gamma$$
$$\angle LDK = 2\alpha - \pi$$

and therefore in directed angles (mod $2\pi$) we have

$$\angle (C'L, B'K) = \angle C'L'C + \angle CLD + \angle LDK + \angle DKB + \angle BK'B' = 2\alpha + 2\beta + 2\gamma - \pi = \pi$$

and hence $C'L$ and $B'K$ are parallel and in opposite directions, i.e. $C'L$ and $KB'$ are in the same direction, as claimed.

**Comment.** While not necessary for the final solution, the following related observation motivates how the fact that $H$ is the midpoint of $KL$ (and therefore $B'KC'L$ is a parallelogram) was first conjectured. We have $AB' = AD = AC'$ by the reflections, i.e. $B'AC'$ is an isosceles triangle with $H$ being the midpoint of the base. Thus $AH$ is the median, altitude and angle bisector in $B'AC'$, thus $\angle B'AK + \angle KAH = \angle HAL + \angle LAC'$. Since from the reflections we also have $\angle B'AK = \angle KAD$ and $\angle DAL = \angle LAC'$ it follows that $\angle HAL + \angle KAD$ and $\angle KAH = \angle DAL$. Since $D$ is the symmedian point in $AKL$, the angle conjugation implies $AH$ is the median line of $KL$. Thus, if $H$ is indeed on $KL$ (as the problem assures us), it can only be the midpoint of $KL$.

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**Solution 6.** There are a number of “phantom point” arguments which define $K'$ and $L'$ in terms of angles and then deduce that these points are actually $K$ and $L$.

Note: In these solutions it is necessary to show that $K$ and $L$ are uniquely determined by the conditions of the problem. One example of doing this is the following:

To prove uniqueness of $K$ and $L$, let us consider that there exist two other points $K'$ and $L'$ that satisfy the same properties ($K'$ on $AB$ and $L'$ on $AC$ such that $DK'$ and $DL'$ are tangent to the circle $AK'L'$).

Then, we have that $DK = DL$ and $DK' = DL'$. We also have that $\angle KDL = \angle K'DL' = \pi - 2\angle A$. Hence, we deduce $\angle KDK' = \angle KDL - \angle K'DL = \angle K'DL' - \angle KDL = \angle LDL'$. Thus we have that $\triangle KDK' \equiv \triangle LDL'$, so we deduce $\angle DKA = \angle DKK' = \angle DLL' = \pi - \angle ALD$. This implies that $AKDL$ is concyclic, which is clearly a contradiction since $\angle KAL + \angle KDL = \pi - \angle BAC$.

---

**Solution 7.** We will use the usual complex number notation, where we will use a capital letter (like $Z$) to denote the point associated to a complex number (like $z$). Consider $\triangle AKL$ on the unit circle. So, we have $a \cdot \bar{a} = k \cdot \bar{k} = l \cdot \bar{l} = 1$ As point $D$ is the intersection of the tangents to the unit circle at $K$ and $L$, we have that

$$d = \frac{2kl}{k + l} \quad \text{and} \quad \bar{d} = \frac{2}{k + l}$$

Defining $B$ as the foot of the perpendicular from $D$ on the line $AK$, and $C$ as the foot of the perpendicular from $D$ on the line $AL$, we have the formulas:

$$b = \frac{1}{2} \left( d + \frac{(a - k)\bar{d} + \bar{a}k - a\bar{k}}{\bar{a} - k} \right)$$
\[ c = \frac{1}{2} \left( d + \frac{(a-l)d + \bar{a}l - a\bar{l}}{\bar{a} - l} \right) \]

Simplying these formulas, we get:

\[ b = \frac{1}{2} \left( d + \frac{(a-k)\frac{2}{k+l} + \frac{k}{a} - \frac{k}{l}}{\frac{1}{a} - \frac{1}{l}} \right) = \frac{1}{2} \left( d + \frac{\frac{2(a-k)}{k+l} + \frac{k^2-a^2}{ak}}{\frac{k-a}{ak}} \right) \]

\[ b = \frac{1}{2} \left( \frac{2kl}{k+l} - \frac{2ak}{k+l} + (a+k) \right) = \frac{k(l-a)}{k+l} + \frac{1}{2}(k+a) \]

\[ c = \frac{1}{2} \left( d + \frac{(a-l)\frac{2}{k+l} + \frac{l}{a} - \frac{l}{k}}{\frac{1}{a} - \frac{1}{k}} \right) = \frac{1}{2} \left( d + \frac{\frac{2(a-l)}{k+l} + \frac{l^2-a^2}{al}}{\frac{l-a}{al}} \right) \]

\[ c = \frac{1}{2} \left( \frac{2kl}{k+l} - \frac{2al}{k+l} + (a+l) \right) = \frac{l(k-a)}{k+l} + \frac{1}{2}(l+a) \]

Let \( O \) be the the circumcenter of triangle \( \triangle ABC \). As \( AD \) is the diameter of this circle, we have that:

\[ o = \frac{a+d}{2} \]

Defining \( H \) as the orthocentre of the \( \triangle ABC \), we get that:

\[ h = a + b + c - 2 \cdot o = a + \left( \frac{k(l-a)}{k+l} + \frac{1}{2}(k+a) \right) + \left( \frac{l(k-a)}{k+l} + \frac{1}{2}(l+a) \right) - (a+d) \]

\[ h = a + \frac{2kl}{k+l} - \frac{a(k+l)}{k+l} + \frac{1}{2}k + \frac{1}{2}l + a - \left( a + \frac{2kl}{k+l} \right) \]

\[ h = \frac{1}{2}(k+l) \]

Hence, we conclude that \( H \) is the midpoint of \( KL \), so \( H, K, L \) are collinear.

---

**Solution 8.** Let us employ the barycentric coordinates. Set \( A(1,0,0), K(0,1,0), L(0,0,1) \).

The tangent at \( K \) of \( (AKL) \) is \( a^2z + c^2x = 0 \), and the tangent of of \( L \) at \( (AKL) \) is \( a^2y + b^2x = 0 \). Their intersection is

\[ D(-a^2:b^2:c^2). \]

Since \( B \in AK \), we can let \( B(1-t,t,0) \). Solving for \( \overrightarrow{AB} \cdot \overrightarrow{BD} = 0 \) gives

\[ t = \frac{3b^2 + c^2 - a^2}{2(b^2 + c^2 - a^2)} \implies B = \left( \frac{-a^2 - b^2 + c^2}{2(b^2 + c^2 - a^2)}, \frac{-a^2 + 3b^2 + c^2}{2(b^2 + c^2 - a^2)}, 0 \right). \]

Likewise, \( C \) has the coordinate

\[ C = \left( \frac{-a^2 + b^2 - c^2}{2(b^2 + c^2 - a^2)}, \frac{-a^2 + b^2 + 3c^2}{2(b^2 + c^2 - a^2)}, \right). \]
The altitude from $B$ for triangle $ABC$ is
\[-b^2 \left( x - z - \frac{-a^2 - b^2 + c^2}{2(b^2 + c^2 - a^2)} \right) + (c^2 - a^2) \left( y - \frac{-a^2 + 3b^2 + c^2}{2(b^2 + c^2 - a^2)} \right) = 0.\]

Also the altitude from $C$ for triangle $ABC$ is
\[-c^2 \left( x - y - \frac{-a^2 + b^2 - c^2}{2(b^2 + c^2 - a^2)} \right) + (a^2 - b^3) \left( z - \frac{-a^2 + b^2 + 3c^2}{2(b^2 + c^2 - a^2)} \right) = 0.\]

The intersection of these two altitudes, which is the orthocenter of triangle $ABC$, has the barycentric coordinate
\[H = (0, 1/2, 1/2),\]
which is the midpoint of the segment $KL$. 

\[\square\]
**Problem 3.** Let $k$ be a positive integer. Lexi has a dictionary $\mathcal{D}$ consisting of some $k$-letter strings containing only the letters $A$ and $B$. Lexi would like to write either the letter $A$ or the letter $B$ in each cell of a $k \times k$ grid so that each column contains a string from $\mathcal{D}$ when read from top-to-bottom and each row contains a string from $\mathcal{D}$ when read from left-to-right.

What is the smallest integer $m$ such that if $\mathcal{D}$ contains at least $m$ different strings, then Lexi can fill her grid in this manner, no matter what strings are in $\mathcal{D}$?

**Solution.** We claim the minimum value of $m$ is $2^k - 1$.

Firstly, we provide a set $S$ of size $2^k - 1 - 1$ for which Lexi cannot fill her grid. Consider the set of all length-$k$ strings containing only $A$s and $B$s which end with a $B$, and remove the string consisting of $k$ $B$s. Clearly there are 2 independent choices for each of the first $k - 1$ letters and 1 for the last letter, and since exactly one string is excluded, there must be exactly $2^k - 1 - 1$ strings in this set.

Suppose Lexi tries to fill her grid. For each row to have a valid string, it must end in a $B$. But then the right column would necessarily contain $k$ $B$s, and not be in our set. Thus, Lexi cannot fill her grid with our set, and we must have $m \geq 2^k - 1$.

Now, consider any set $S$ with at least $2^k - 1$ strings. Clearly, if $S$ contained either the uniform string with $k$ $A$s or the string with $k$ $B$s, then Lexi could fill her grid with all of the relevant letters and each row and column would contain that string.

Consider the case where $S$ contains neither of those strings. Among all $2^k$ possible length-$k$ strings with $A$s and $B$s, each has a complement which corresponds to the string with $B$s in every position where first string had $A$s and vice-versa. Clearly, the string with all $A$s is paired with the string with all $B$s. We may assume that we do not take the two uniform strings and thus applying the pigeonhole principle to the remaining set of strings, we must have two strings which are complementary.

Let this pair of strings be $\ell, \ell' \in S$ in some order. Define the set of indices $J$ corresponding to the $A$s in $\ell$ and thus the $B$s in $\ell'$, and all other indices (not in $J$) correspond to $B$s in $\ell$ (and thus $A$s in $\ell'$). Then, we claim that Lexi puts an $A$ in the cell in row $r$, column $c$ if $r, c \in J$ or $r, c \notin J$, and a $B$ otherwise, each row and column contains a string in $S$.

We illustrate this with a simple example: If $k = 6$ and we have that $AAABAB$ and $BBBABA$ are both in the dictionary, then Lexi could fill the table as follows:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>B</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
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<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td></td>
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<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
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</tr>
<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
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</tr>
<tr>
<td>B</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td></td>
</tr>
</tbody>
</table>

Suppose we are looking at row $i$ or column $i$ for $i \in J$. Then by construction the string in this row/column contains $A$s at indices $k$ with $k \in J$ and $B$s elsewhere, and thus is precisely $\ell$. Suppose instead we are looking at row $i$ or column $i$ for $i \notin J$. Then again
by construction the string in this row/column contains $A$s at indices $k$ with $k \notin J$ and $B$s elsewhere, and thus is precisely $\ell'$. So each row and column indeed contains a string in $S$.

Thus, for any $S$ with $|S| \geq 2^{k-1}$, Lexi can definitely fill the grid appropriately. Since we know $m \geq 2^{k-1}$, $2^{k-1}$ is the minimum possible value of $m$ as claimed. \[\square\]
**Problem 4.** Turbo the snail sits on a point on a circle with circumference 1. Given an infinite sequence of positive real numbers $c_1, c_2, c_3, \ldots$, Turbo successively crawls distances $c_1, c_2, c_3, \ldots$ around the circle, each time choosing to crawl either clockwise or counterclockwise.

For example, if the sequence $c_1, c_2, c_3, \ldots$ is $0.4, 0.6, 0.3, \ldots$, then Turbo may start crawling as follows:

![Crawling Diagram]

Determine the largest constant $C > 0$ with the following property: for every sequence of positive real numbers $c_1, c_2, c_3, \ldots$ with $c_i < C$ for all $i$, Turbo can (after studying the sequence) ensure that there is some point on the circle that it will never visit or crawl across.

**Solution 1.** The largest possible $C$ is $C = \frac{1}{2}$.

For $0 < C \leq \frac{1}{2}$, Turbo can simply choose an arbitrary point $P$ (different from its starting point) to avoid. When Turbo is at an arbitrary point $A$ different from $P$, the two arcs $AP$ have total length 1; therefore, the larger of the two the arcs (or either arc in case $A$ is diametrically opposite to $P$) must have length $\geq \frac{1}{2}$. By always choosing this larger arc (or either arc in case $A$ is diametrically opposite to $P$), Turbo will manage to avoid the point $P$ forever.

For $C > \frac{1}{2}$, we write $C = \frac{1}{2} + a$ with $a > 0$, and we choose the sequence

\[
\frac{1}{2}, \quad \frac{1 + a}{2}, \quad \frac{1}{2}, \quad \frac{1 + a}{2}, \quad \frac{1}{2}, \quad \ldots
\]

In other words, $c_i = \frac{1}{2}$ if $i$ is odd and $c_i = \frac{1 + a}{2} < C$ when $i$ is even. We claim Turbo must eventually visit all points on the circle. This is clear when it crawls in the same direction two times in a row; after all, we have $c_i + c_{i+1} > 1$ for all $i$. Therefore, we are left with the case that Turbo alternates crawling clockwise and crawling counterclockwise. If it, without loss of generality, starts by going clockwise, then it will always crawl a distance $\frac{1}{2}$ clockwise followed by a distance $\frac{1 + a}{2}$ counterclockwise. The net effect is that it crawls a distance $\frac{a}{2}$ counterclockwise. Because $\frac{a}{2}$ is positive, there exists a positive integer $N$ such that $\frac{a}{2} \cdot N > 1$. After $2N$ crawls, Turbo will have crawled a distance $\frac{a}{2}$ counterclockwise $N$ times, therefore having covered a total distance of $\frac{a}{2} \cdot N > 1$, meaning that it must have crawled over all points on the circle.

**Note:** Every sequence of the form $c_i = x$ if $i$ is odd, and $c_i = y$ if $i$ is even, where $0 < x, y < C$, such that $x + y \geq 1$, and $x \neq y$ satisfies the conditions with the same argument. There might be even more possible examples.
Solution 2. Alternative solution (to show that \( C \leq \frac{1}{2} \))

We consider the following related problem:

We assume instead that the snail Chet is moving left and right on the real line. Find the size \( M \) of the smallest (closed) interval, that we cannot force Chet out of, using a sequence of real numbers \( d_i \) with \( 0 < d_i < 1 \) for all \( i \).

Then \( C = 1/M \). Indeed if for every sequence \( c_1, c_2, \ldots \), with \( c_i < C \) there exists a point that Turbo can avoid, then the circle can be cut open at the avoided point and mapped to an interval of size \( M \) such that Chet can stay inside this interval for any sequence of the form \( c_1/C, c_2/C, \ldots \), see Figure 5. However, all sequences \( d_1, d_2, \ldots \) with \( d_i < 1 \) can be written in this form. Similarly if for every sequence \( d_1, d_2, \ldots \), there exists an interval of length smaller or equal \( M \) that we cannot force Chet out of, this projects to a subset of the circle, that we cannot force Turbo out of using any sequence of the form \( d_1/M, d_2/M, \ldots \). These are again exactly all the sequences with elements in \( [0, C) \).

![Figure 5: Chet and Turbo equivalence](image)

Claim: \( M \geq 2 \).

Proof. Suppose not, so \( M < 2 \). Say \( M = 2 - 2\varepsilon \) for some \( \varepsilon > 0 \) and let \([-1 + \varepsilon, 1 - \varepsilon]\) be a minimal interval, that Chet cannot be forced out of. Then we can force Chet arbitrarily close to \( \pm(1 - \varepsilon) \). In particular, we can force Chet out of \([-1 + \frac{4}{3}\varepsilon, 1 - \frac{4}{3}\varepsilon]\) by minimality of \( M \). This means that there exists a sequence \( d_1, d_2, \ldots \) for which Chet has to leave \([-1 + \frac{4}{3}\varepsilon, 1 - \frac{4}{3}\varepsilon]\), which means he ends up either in the interval \([-1 + \varepsilon, -1 + \frac{4}{3}\varepsilon]\) or in the interval \((1 - \frac{4}{3}\varepsilon, 1 - \varepsilon]\).

Now consider the sequence,

\[
d_1, \ 1 - \frac{7}{6}\varepsilon, \ 1 - \frac{2}{3}\varepsilon, \ 1 - \frac{2}{3}\varepsilon, \ 1 - \frac{7}{6}\varepsilon, \ 1 - \frac{2}{3}\varepsilon, \ 1 - \frac{7}{6}\varepsilon, \ 1 - \frac{2}{3}\varepsilon, \ 1 - \frac{7}{6}\varepsilon, \ 1 - \frac{2}{3}\varepsilon, \ 1 - \frac{7}{6}\varepsilon, \ \ldots
\]

obtained by adding the sequence \( 1 - \frac{7}{6}\varepsilon, \ 1 - \frac{2}{3}\varepsilon, \ 1 - \frac{2}{3}\varepsilon, \ 1 - \frac{7}{6}\varepsilon \) in between every two steps. We claim that this sequence forces Chet to leave the larger interval \([-1 + \varepsilon, 1 - \varepsilon]\). Indeed no two consecutive elements in the sequence \( 1 - \frac{7}{6}\varepsilon, \ 1 - \frac{2}{3}\varepsilon, \ 1 - \frac{2}{3}\varepsilon, \ 1 - \frac{7}{6}\varepsilon \) can...
have the same sign, because the sum of any two consecutive terms is larger than $2 - 2\varepsilon$ and Chet would leave the interval $[-1 + \varepsilon, 1 - \varepsilon]$. It follows that the $(1 - \frac{7}{6}\varepsilon)$’s and the $(1 - \frac{2}{3}\varepsilon)$’s cancel out, so the position after $d_k$ is the same as before $d_{k+1}$. Hence, the positions after each $d_k$ remain the same as in the original sequence. Thus, Chet is also forced to the boundary in the new sequence.

If Chet is outside the interval $[-1 + \frac{1}{3}\varepsilon, 1 - \frac{4}{3}\varepsilon]$, then Chet has to move $1 - \frac{7}{6}\varepsilon$ towards 0, and ends in $[-\frac{1}{6}\varepsilon, \frac{1}{6}\varepsilon]$. Chet then has to move by $1 - \frac{2}{3}\varepsilon$, which means that he has to leave the interval $[-1 + \varepsilon, 1 - \varepsilon]$. Indeed the absolute value of the final position is at least $1 - \frac{5}{6}\varepsilon$. This contradicts the assumption, that we cannot force Chet out of $[-1 + \varepsilon, 1 - \varepsilon]$. Hence $M \geq 2$ as needed.
**Problem 5.** We are given a positive integer \(s \geq 2\). For each positive integer \(k\), we define its twist \(k'\) as follows: write \(k\) as \(as + b\), where \(a, b\) are non-negative integers and \(b < s\), then \(k' = bs + a\). For the positive integer \(n\), consider the infinite sequence \(d_1, d_2, \ldots\) where \(d_i = n\) and \(d_{i+1}\) is the twist of \(d_i\) for each positive integer \(i\).

Prove that this sequence contains 1 if and only if the remainder when \(n\) is divided by \(s^2 - 1\) is either 1 or \(s\).

---

**Solution 1.** First, we consider the difference \(k - k''\). If \(k = as + b\) as in the problem statement, then \(k' = bs + a\). We write \(a = ls + m\) with \(m, l\) non-negative numbers and \(m \leq s - 1\). This gives \(k'' = ms + (b + l)\) and hence \(k - k'' = (a - m)s - l = l(s^2 - 1)\).

We conclude

**Fact 1.1.** \(k \geq k''\) for every every \(k \geq 1\)

**Fact 1.2.** \(s^2 - 1\) divides the difference \(k - k''\).

Fact 1.2 implies that the sequences \(d_1, d_3, d_5, \ldots\) and \(d_2, d_4, d_6, \ldots\) are constant modulo \(s^2 - 1\). Moreover, Fact 1.1 says that the sequences are (weakly) decreasing and hence eventually constant. In other words, the sequence \(d_1, d_2, d_3, \ldots\) is 2-periodic modulo \(s^2 - 1\) (from the start) and is eventually 2-periodic.

Now, assume that some term in the sequence is equal to 1. The next term is equal to \(1' = s\) and since the sequence is 2-periodic from the start modulo \(s^2 - 1\), we conclude that \(d_1\) is either equal to 1 or \(s\) modulo \(s^2 - 1\). This proves the first implication.

To prove the other direction, assume that \(d_1\) is congruent to 1 or \(s\) modulo \(s^2 - 1\). We need the observation that once one of the sequences \(d_1, d_3, d_5, \ldots\) or \(d_2, d_4, d_6, \ldots\) stabilises, then their value is less than \(s^2\). This is implied by the following fact.

**Fact 1.3.** If \(k = k''\), then \(k = k'' < s^2\).

*Proof.* We use the expression for \(k - k''\) found before. If \(k = k''\), then \(l = 0\), and so \(k'' = ms + b\). Both \(m\) and \(b\) are reminders after division by \(s\), so they are both \(\leq s - 1\). This gives \(k'' \leq (s - 1)s + (s - 1) < s^2\). \(\square\)

Using Fact 1.2, it follows that the sequence \(d_1, d_3, d_5, \ldots\) is constant to 1 or \(s\) modulo \(s^2 - 1\) and stabilises to 1 or \(s\) by Fact 1.3. Since \(s' = 1\), we conclude that the sequence contains a 1. \(\square\)

**Solution 2.** We make a number of initial observations. Let \(k\) be a positive integer.

**Fact 2.1.** If \(k \geq s^2\), then \(k' < k\).

*Proof.* Write \(k = as + b\), as in the problem statement. If \(k \geq s^2\), then \(a \geq s\) because \(b < s\). So, \(k' = bs + a \leq (s - 1)s + a \leq as \leq as + b = k\). Moreover, we cannot have equality since that would imply \(s - 1 = b = 0\). \(\square\)

**Fact 2.2.** If \(k \leq s^2 - 1\), then \(k' \leq s^2 - 1\) and \(k'' = k\).
Proof. Write \( k = as + b \), as in the problem statement. If \( k < s^2 \), then it must hold \( 1 \leq a, b < s \), hence \( k' = bs + a < s^2 \) and \( k'' = (bs + a)' = as + b = k \). \( \square \)

Fact 2.3. We have \( k' \equiv sk \pmod{s^2 - 1} \) (or equivalently \( k \equiv sk' \pmod{s^2 - 1} \)).

Proof. We write \( k = as + b \), as in the problem statement. Now, \( sk - k' = s(as + b) - (bs + a) = a(s^2 - 1) \equiv 0 \pmod{s^2 - 1} \), as desired. \( \square \)

Combining Facts 2.1 and 2.2, we find that the sequence \( d_1, d_2, d_3 \ldots \) is eventually periodic with period 2, starting at the first value less than \( s^2 \). From Fact 2.3, it follows that

\[
k'' = sk' \equiv s^2k \equiv k \pmod{s^2 - 1}
\]

and hence the sequence is periodic modulo \( s^2 - 1 \) from the start with period 2.

Now, if the sequence contains 1, the sequence eventually alternates between 1 and \( s \) since the twist of 1 is \( s \) and vice versa. Using periodicity modulo \( s^2 - 1 \), we must have \( n \equiv 1, s \pmod{s^2 - 1} \). Conversely, if \( n \equiv 1, s \pmod{s^2 - 1} \) then the eventual period must contain at least one value congruent to either 1 or \( s \) modulo \( s^2 - 1 \). Since these values must be less than \( s^2 \), this implies that the sequence eventually alternates between 1 and \( s \), showing that it contains a 1. \( \square \)

Solution 3. We give an alternate proof of the direct implication: if the sequence contains a 1, then the first term is 1 or \( s \) modulo \( s^2 - 1 \). We prove the following fact, which is a combination of Facts 2.1 and 2.3.

Fact 3.1. For all \( k \geq s^2 \), we have \( (k - s^2 + 1)' \in \{k', k' - s^2 + 1\} \).

Proof. We write \( k = as + b \), as in the problem statement. Since \( k \geq s^2 \), we have \( a \geq s \). If \( b < s - 1 \), then

\[
(k - s^2 + 1)' = ((a - s)s + (b + 1))' = (b + 1)s + (a - s) = bs + a = k'.
\]

On the other hand, if \( b = s - 1 \), then

\[
(k - s^2 + 1)' = ((a - s + 1)s + 0)' = 0s + (a - s + 1) = a - s + 1 = k' - s^2 + 1.
\]

Now assume \( n \geq s^2 \) and the sequence \( d_1, d_2, \ldots \) contains a 1. Denote by \( e_1, e_2, \ldots \) the sequence constructed as in the problem statement, but with initial value \( e_1 = n - s^2 + 1 \). Using the above fact, we deduce that \( e_i \equiv d_i \pmod{s^2 - 1} \) and \( e_i \leq d_i \) for all \( i \geq 1 \) by induction on \( i \). Hence, the sequence \( e_1, e_2, \ldots \) also contains a 1.

Since the conclusion we are trying to reach only depends on the residue of \( d_1 \) modulo \( s^2 - 1 \), we conclude that without loss of generality we can assume \( n < s^2 \).
Using Fact 2.2, it now follows that the sequence \( d_1, d_2, \ldots \) is periodic with period two. Since \( 1 \) and \( s \) are twists of each other, it follows that if this sequence contains a 1, it must be alternating between 1 and \( s \). Hence, \( d_1 \equiv 1, s \pmod{s^2 - 1} \) as desired.

For the other direction we can make a similar argument, observing that the second of the two cases in the proof of Fact 3.1 can only apply to finitely many terms of the sequence \( d_1, d_2, d_3, \ldots \), allowing us to also go the other way.

\[ \square \]

**Solution 4.** First assume that \( d_k = 1 \) for some \( k \). Let \( k \) be the smallest such index. If \( k = 1 \) then \( n = 1 \), so we may assume \( k \geq 2 \).

Then \( d_{k-1} = as + b \) for some non-negative integers \( a, b \) satisfying \( b < s \) and \( bs + a = 1 \). The only solution is \( b = 0, a = 1 \), so \( d_{k-1} = s \). So, if \( k = 2 \), then \( n = s \), so we may assume \( k \geq 3 \).

Then there exist non-negative integers \( c, d \) satisfying \( d_{k-2} = cs + d \), \( d < s \) and \( ds + c = s \). We have two solutions: \( d = 0, c = s \) and \( d = 1, c = 0 \). However, in the second case we get \( d_{k-2} = 1 \), which contradicts the minimality of \( k \). Hence, \( d_{k-2} = s^2 \). If \( k = 3 \), then \( n = d_1 = s^2 \), which gives remainder 1 when divided by \( s^2 - 1 \).

Assume now that \( k \geq 4 \). We will show that for each \( m \in \{3, 4, \ldots, k-1\} \) there exist \( b_1, b_2, \ldots, b_{m-2} \in \{0, 1, \ldots, s-1\} \) such that

\[ d_{k-m} = s^m - \sum_{i=1}^{m-2} b_i \left( s^{m-i} - s^{m-i-2} \right). \]  

(9)

We will prove this equality by induction on \( m \). If \( m = 3 \), then \( d_{k-3} = a_1s + b_1 \) for some non-negative integers \( a_1, b_1 \) satisfying \( b_1 < s \) and \( b_1s + a_1 = d_{k-2} = s^2 \). Then \( a_1 = s^2 - b_1s \), so \( d_{k-3} = s^3 - b_1(s^2 - 1) \), which proves (9) for \( m = 3 \).

Assume that (9) holds for some \( m \) and consider \( d_{k-(m+1)} \). There exist non-negative integers \( a_{m-1}, b_{m-1} \) such that \( d_{k-(m+1)} = a_{m-1}s + b_{m-1}, b_{m-1} < s \) and \( d_{k-m} = b_{m-1}s + a_{m-1} \). Using the inductive assumption we get

\[ a_{m-1} = d_{k-m} - b_{m-1}s = s^m - \sum_{i=1}^{m-2} b_i \left( s^{m-i} - s^{m-i-2} \right) - b_{m-1}s, \]

therefore

\[ d_{k-(m+1)} = a_{m-1}s + b_{m-1} = s^{m+1} - \sum_{i=1}^{m-2} b_i \left( s^{m-i} - s^{m-i-2} \right) s - b_{m-1}s^2 + b_{m-1} \]

\[ = s^{m+1} - \sum_{i=1}^{m-1} b_i \left( s^{m+1-i} - s^{m-i-1} \right), \]

which completes the proof of (9). In particular, for \( m = k - 1 \) we get

\[ d_1 = s^{k-1} - \sum_{i=1}^{k-3} b_i \left( s^{k-i-1} - s^{k-i-3} \right). \]

The above sum is clearly divisible by \( s^2 - 1 \), and it is clear that the remainder of \( s^{k-1} \) when divided by \( s^2 - 1 \) is 1 when \( k \) is odd, and \( s \) when \( k \) is even. It follows that the remainder when \( n = d_1 \) is divided by \( s^2 - 1 \) is either 1 or \( s \).
To prove the other implication, assume that \( n \) gives remainder 1 or \( s \) when divided by \( s^2 - 1 \). If \( n \in \{1, s, s^2\} \), then one of the numbers \( d_1, d_2 \) and \( d_3 \) is 1. We therefore assume that \( n > s^2 \). Since the reminder when a power of \( s \) is divided by \( s^2 - 1 \) is either 1 or \( s \), there exists a positive integer \( m \) such that \( s^m - n \) is non-negative and divisible by \( s^2 - 1 \). By our assumption \( m \geq 3 \). We also take the smallest such \( m \), so that \( m \) is a positive integer \( m \) such that \( s^m - n \) is non-negative and divisible by \( s^2 - 1 \). We therefore assume that \( n > s^2 \). Since the reminder when a power of \( s \) is divided by \( s^2 - 1 \) is either 1 or \( s \), there exists a positive integer \( m \) such that \( s^m - n \) is non-negative and divisible by \( s^2 - 1 \).

By our assumption \( m \geq 3 \). We also take the smallest such \( m \), so that \( n > s^m - 2 \). The quotient \( s^m - n \) is therefore smaller than \( s^m - 2 \), so there exist \( b_1, \ldots, b_{m-2} \in \{0, 1, \ldots, s - 1\} \) such that \( s^m - n = \sum_{i=1}^{m-2} b_i s^{i-1} \). It follows that

\[
n = s^m - \sum_{i=1}^{m-2} b_i \left( s^{i+1} - s^{i-1} \right).
\]

We now show that

\[
d_j = s^{m+1-j} - \sum_{i=1}^{m-1-j} b_i \left( s^{i+1} - s^{i-1} \right)
\]

for \( j = 1, 2, \ldots, m - 2 \) by induction on \( j \). For \( j = 1 \) this follows from \( d_1 = n \). Assume now that (10) holds for some \( j < m - 2 \). Then

\[
d_j = \left( s^{m-j} - \sum_{i=2}^{m-1-j} b_i \left( s^{i} - s^{i-2} \right) \right) s + b_1.
\]

As \( d_j \) is positive and \( b_1 \in \{0, 1, \ldots, s - 1\} \), the expression \( s^{m-j} - \sum_{i=2}^{m-1-j} b_i \left( s^{i} - s^{i-2} \right) - b_1 s \) has to be non-negative, so we can compute the twist of \( d_j \) as

\[
d_{j+1} = b_1 s + s^{m-j} - \sum_{i=2}^{m-1-j} b_i \left( s^{i} - s^{i-2} \right) - b_1 s = s^{m-j} - \sum_{i=1}^{m-2-j} b_i \left( s^{i+1} - s^{i-1} \right),
\]

which finishes the induction.

Now we use (10) for \( j = m - 2 \) and get \( d_{m-2} = s^3 - b_1 (s^2 - 1) = (s^2 - b_1 s) + b_1 \). Then \( d_{m-1} = b_1 s + s^2 - b_1 s = s^2 = s \cdot s + 0, d_m = 0 \cdot s + s = s = 1 \cdot s + 0 \) and \( d_{m+1} = 0 \cdot s + 1 = 1 \). \( \square \)
Problem 6. Let $ABC$ be a triangle with circumcircle $\Omega$. Let $S_b$ and $S_c$ respectively denote the midpoints of the arcs $AC$ and $AB$ that do not contain the third vertex. Let $N_a$ denote the midpoint of arc $BAC$ (the arc $BC$ containing $A$). Let $I$ be the incentre of $ABC$. Let $\omega_b$ be the circle that is tangent to $AB$ and internally tangent to $\Omega$ at $S_b$, and let $\omega_c$ be the circle that is tangent to $AC$ and internally tangent to $\Omega$ at $S_c$. Show that the line $IN_a$, and the line through the intersections of $\omega_b$ and $\omega_c$, meet on $\Omega$.

The incentre of a triangle is the centre of its incircle, the circle inside the triangle that is tangent to all three sides.

Solution 1. Part I: First we show that $A$ lies on the radical axis of $\omega_b$ and $\omega_c$.

We first note that the line through the intersections of two circles is the radical line of the two circles. Let the tangents to $\Omega$ at $S_b$ and $S_c$ intersect at $T$. Clearly $T$ is on the radical axis of $\omega_b$ and $\omega_c$ (and in fact is the radical centre of $\omega_b, \omega_c$ and $\Omega$).

We next show that $A$ lies on the radical axis of $\omega_b$ and $\omega_c$. Let $P_b$ denote the point of tangency of $\omega_b$ and $AB$, and let $P_c$ denote the point of tangency of $\omega_c$ and $AC$. Furthermore, let $U$ be the intersection of the tangent to $\Omega$ at $S_b$ with the line $AB$, and let $V$ be the intersection of the tangent to $\Omega$ at $S_c$ with the line $AC$. Then $TVAU$ is parallelogram. Morover, due to equality of tangent segments we have $|US_b| = |UP_b|$, $|VP_c| = |VS_c|$ and $|TS_b| = |TS_c|$. It follows that

$$|AP_b| = |UP_b| - |UA| = |US_b| - |TV| = |TS_b| - |TU| - |TV| = |TS_b| - |TV| - |TU| = |VS_c| - |AV| = |VP_c| - |VA| = |AP_c|. \quad (11)$$
But $|AP_b|$, $|AP_c|$ are exactly the square roots of powers of $A$ with respect to $\omega_b$ and $\omega_c$, hence $A$ is indeed on their radical axis.

Thus, the radical axis of $\omega_b, \omega_c$ is $AT$.

Part II: Consider the triangle $AS_bS_c$. Note that since $T$ is the intersection of the tangents at $S_b$ and $S_c$ to the circumcircle of $AS_bS_c$, it follows that $AT$ is the symmedian of $A$ in this triangle. Let $X$ denote the second intersection of the symmedian $AT$ with $\Omega$. We wish to show that $X$ is also on $IN_a$.

Note that $AN_a$ is the external angle bisector of angle $A$, and therefore it is parallel to $S_bS_c$. Let $M$ denote the midpoint of $S_bS_c$, and let $Y$ be the second intersection of $AM$ with $\Omega$. Since in $AS_bS_c$, $AXT$ is the symmedian and $AMY$ is the median, it follows that $XY$ is also parallel to $S_bS_c$. Thus, reflecting in the perpendicular bisector of $S_bS_c$ sends the line $AMY$ to line $N_aMX$.

Next, consider the quadrilateral $AS_bIS_c$. From the trillium theorem we have $|S_bA| = |S_bI|$ and $|S_cA| = |S_cI|$, thus the quadrilateral is a kite, from which it follows that the reflection of the line $AM$ in $S_bS_c$ is the line $IM$. But previously we have seen that this is also the line $N_aMX$. Thus $M, I, N_a$ and $X$ are collinear, as we wanted to show.

**Solution 2.** This is a variation of Solution 1 which avoids the theory of the symmedian point.

We begin by showing that the radical axis of $\omega_b, \omega_c$ is $AT$ as in Solution 1.

Part II: We introduce the point $S_a$ with the obvious meaning. Observe that the incentre $I$ of $ABC$ is the orthocentre of $S_aS_bS_c$ either because this is well-known, or because of an angle argument that $A$ reflects in $S_bS_c$ to $I$ (and similar results by cyclic change of letters). Therefore $AS_a$ is perpendicular to $S_bS_c$. 

![Figure 7: A reflections argument for Solution 2](image)

Let $M$ denote the midpoint of $S_bS_c$. Then $A$ is the reflection of $S_a$ in the diameter parallel to $S_bS_c$, so the reflection of $A$ in the diameter perpendicular to $S_bS_c$ is $N_a$, the antipode
of $S_a$. Let the reflection of $X$ in $TM$ be $Y$, so $TY$ passes through $N_a$ and is the reflection of $TX$ in $TM$.

Now $S_bS_c$ is the polar line of $T$ with respect to $\Omega$, so $AY$ and $N_aX$ meet on this line, and by symmetry at its midpoint $M$. The line $N_aMX$ is therefore the reflection of the line $YMA$ in $S_bS_c$, and so $N_aMX$ passes through $I$ (the reflection of $A$ in $S_bS_c$).

The triangle $AS_cS_b$ can be taken as generic, and from the argument above we can extract the fact that the symmedian point and the centroid are isogonal conjugates in that triangle.

\[\square\]

\[\textbf{Solution 3.}\] Assume the notation from Solution 1, part I of Solution 1, and let $O$ be the centre of $\Omega$.

\hspace{1cm}

\[\text{Figure 8: Diagram to Solution 3}\]

Part II: As in Solution 1, by the trillelium theorem, $S_cS_b$ bisects $AI$, and since $N_aA \parallel S_bS_c$, then $OT$ is a bisector of $AN_a$. This implies $|MN_a| = |MA| = |MI|$, since $M$ is the midpoint of $S_cS_b$ and lies also on $OT$. Hence, $M$ is the circumcentre of triangle $IAN_a$.

But this triangle has a right angle at $A$ (since $AI$ and $AN_a$ are the inner and outer angle bisector at $A$), hence $M$ lies on $IN_a$.

Again, let $X$ be the second intersection of $TA$ and $\Omega$. By the above, it suffices to prove that $X$ lies on the line $N_aM$. From the power of point $T$ with respect to $\Omega$ we get $|TA| \cdot |TX| = |TS_c|^2$. Since $M$ is the foot of the altitude of right triangle $TS_cO$, we obtain
\[|TS_c|^2 = |TM| \cdot |TO|.\] Hence, \(|TA| \cdot |TX| = |TM| \cdot |TO|\) so the points \(O, M, A, X\) are concyclic. It follows that \(\angle MXA = \angle MOA = \frac{1}{2} \angle Na OA = \angle NaXA\). Hence, \(X\) lies on the line \(NaM\).

**Remark.** To show that \(OMAX\) is cyclic, one can also invert the line \(TAX\) in the circumcircle of the triangle \(ABC\).

**Solution 4.** Part I is done as in solution 1.

![Diagram to Solution 4](image-url)

Part II: as in Solution 1 we show that \(AN_a || SB SC\). In particular, \(\angle NaOT = \angle TOA\). The conclusion of the problem trivially holds if \(|AB| = |AC|\), therefore we assume without loss of generality that \(|AC| > |AB|\). Let \(S_a\) be the midpoint of the arc \(BC\) which does not contain \(A\). Then \(NaSa\) is a diameter, so \(\angle SaBa = \frac{\pi}{2} = \angle OSaT\). We also compute \(\angle BNaSa = \angle BASa = \frac{1}{2}\angle BAC = \frac{1}{2}\angle S_c TS_b = \angle S_a TO\). It follows that the triangles \(TS_cO\) and \(NaBSa\) are similar. In particular,

\[
\frac{|NaB|}{|TS_c|} = \frac{|SaB|}{|OS_c|}.
\]

Next we compute

\[
\angle IS_a B = \angle NaSaB - \angle NaSaI = \angle TOSc - \frac{1}{2} \angle Na OA = \angle TOSc - \angle TOA = \angle AOS_c
\]
and

\[ \angle IBN_a = \angle CBN_a - \angle CBI = \frac{1}{2}(\pi - \angle BN_a C) - \frac{1}{2} \angle CBA = \frac{1}{2} \angle ACB = \angle ACS_c = \angle AS_c T, \]

hence

\[ \angle S_a BI = \frac{\pi}{2} - \angle IBN_a = \frac{\pi}{2} - \angle AS_c T = \angle OS_c A. \tag{14} \]

Together with (13) it follows that the triangles \( IBS_a \) and \( AS_c O \) are similar, so \( \frac{|S_a B|}{|OS_c|} = \frac{|IB|}{|AS_c|} \), and (12) implies \( \frac{|N_a B|}{|TS_c|} = \frac{|IB|}{|AS_c|} \). Consequently, by (14) the triangles \( TS_c A \) and \( N_a BI \) are similar and therefore \( \angle S_c TA = \angle B N_a I \). Now let \( Q \) be the second intersection of \( N_a I \) with \( \Omega \). Then \( \angle B N_a I = \angle B N_a Q = \angle BAQ \), so \( \angle S_c TA = \angle BAQ \). Since \( AB \) is parallel to \( TS_c \), we get \( AQ \parallel TA \), i.e. \( A, T, Q \) are collinear. \qed

**Remark.** After proving similarity of triangles \( TS_c O \) and \( N_a BS_a \) one can use spiral symmetry to show similarity of triangles \( TS_c A \) and \( N_a BI \).

**Solution 5.** Part I: First we show that \( A \) lies on the radical axis between \( \omega_b \) and \( \omega_c \).

Let \( T \) be the radical center of the circumcircle, \( \omega_b \) and \( \omega_c \); then \( TS_c \) and \( TS_b \) are common tangents of the circles, as shown in Figure 5a. Moreover, let \( P_b = AB \cap S_b S_c \) and \( P_c = AC \cap S_b S_c \). The triangle \( TS_c S_b \) is isosceles: \( AB \parallel TS_c \) and \( AC \parallel TS_b \) so

\[ \angle AP_b P_c = \angle TS_c S_b = \angle S_c S_b T = \angle P_b P_c A. \]

From these angles we can see that \( \omega_b \) passes through \( P_b \), \( \omega_c \) passes through \( P_c \), and finally \( AP_b \) and \( AP_c \) are equal tangents to \( \omega_b \) and \( \omega_c \), so \( A \) lies on the radical axis.
Part II. Let the radical axis $TA$ meet the circumcircle again at $X$, let $S_a$ be the midpoint of the arc $BC$ opposite to $A$, and let $XI$ meet the circumcircle again at $N$. (See Figure 2.) For solving the problem, we have prove that $N = N_a$.

The triples of points $A, I, S_a; B, I, S_b$ and $C, I, S_c$ are collinear because they lie on the angle bisectors of the triangle $ABC$.

Notice that the quadrilateral $AS_cXS_b$ is harmonic, because the tangents at $S_b$ and $S_c$, and the line $AX$ are concurrent at $T$. This quadrilateral can be projected (or inverted) to the quadrilateral $S_aCNB$ through $I$. So, $S_aCNB$ also is a harmonic quadrilateral. Due to $S_aB = S_aC$, this implies $NB = NC$, so $N = N_a$. Done.

**Remark.** Instead of mentioning inversion and harmonic quadrilaterals, from the similar triangles $\triangle TS_cA \sim \triangle TXS_c$ and $\triangle TAS_b \sim \triangle TS_bX$ we can get

$$\frac{AS_c}{S_cX} = \frac{AS_b}{S_bX}.$$

Then, we can apply the trigonometric form of Ceva’s theorem to triangle $BCX$

$$\frac{\sin \angle BXN_a}{\sin \angle N_aXC} \cdot \frac{\sin \angle CBS_b}{\sin \angle S_bBX} \cdot \frac{\sin \angle XCS_c}{\sin \angle S_cCB} = \frac{BN_a}{N_aC} \cdot \frac{-CS_b}{S_bX} \cdot \frac{XS_c}{-S_cB} = 1 \cdot \frac{S_bA}{S_bN_a} \cdot \frac{N_aS_c}{S_cB} = 1,$$

so the Cevians $BS_b, CS_c$ and $XN_a$ are concurrent.

**Solution 6.** Part I: First let’s show that this is equivalent to proving that $TA$ and $N_aI$ intersect in $\Omega$.

**Lemma:** Let’s recall that if we have two circles $\omega_1$ and $\omega_2$ which are internally tangent at point $X$ and if we have a line $AB$ tangent to $\omega_2$ at $Y$. Let $M$ be the midpoint of the arc $AB$ not containing $Z$. We have that $Z, Y, M$ are collinear.

![Solution 6: Lemma](image)

Let $P_b = AB \cap \omega_b$ and $P_c = AC \cap \omega_c$. We can notice by the lemma that $S_b, P_b$ and $S_c$ are collinear, and similarly $S_c, P_c$ and $S_b$ are also collinear. Therefore $S_c, P_b, P_c$, and $S_b$ are collinear, and since $\angle AP_bP_c = \frac{\angle ABC}{2} + \frac{\angle AEC}{2} = \angle AP_cP_b$ then $AP_b = AP_c$ so $A$ is on the radical axis of $\omega_b$ and $\omega_c$. Let $T$ be the intersection of the tangent lines of $\Omega$ through $S_c$ and $S_b$. Since $TS_c = TS_b$ then $AT$ is the radical axis between $\omega_b$ and $\omega_c$.

Part II: $TA$ and $N_aI$ intersect in $\Omega$.  

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Let $\omega_a$ the $A$-mixtilinear incircle (that is, the circle internally tangent to $\Omega$, and tangent to both $AB$ and $AC$), and let $X = \Omega \cap \omega_a$. It is known that $N_a, I, X$ are collinear.

Let $M_c$ and $M_b$ be the tangent points of $\omega_A$ to $AB$ and $AC$ respectively, then by the lemma $X, M_c, S_c$ are collinear and $X, M_b, S_b$ are collinear. We can see that $S_c T S_b$ and $M_c A M_b$ are homothetic with respect to $X$; therefore $T$ and $A$ are homothetic with respect to $X$, implying that $T, A, X$ are collinear.

Figure 10: Diagram to Solution 6