Solutions of EGMO 2021

Problem 1. According to Anna, the number 2021 is *fantabulous*. She states that if any element of the set $\{m, 2m+1, 3m\}$ is fantabulous for a positive integer m, then they are all fantabulous. Is the number 2021^{2021} fantabulous?

(Australia, Angelo Di Pasquale)

Answer: Yes

Solution 1.

Consider the sequence of positive integers $m, 3m, 6m+1, 12m+3, 4m+1, 2m$. Since each number in the sequence is fantabulous if and only if the next one is, we deduce that m is fantabulous if and only if $2m$ is fantabulous.

Combined with the fact that m is fantabulous if and only if $2m+1$ is fantabulous, this implies that $m > 1$ is fantabulous if and only if $f(m) = \left\lfloor \frac{m}{2} \right\rfloor$ is fantabulous. We can apply f sufficiently many times to any positive integer n to conclude that n is fantabulous if and only if 1 is fantabulous. Therefore, the fact that 2021 is fantabulous implies that 1 is fantabulous, which in turn implies that 2021^{2021} is fantabulous.

Solution 2.

Let $m > 1$ be a fantabulous number. Note that at least one of the following four cases must hold.

■ *Case 1.* The number m is odd;
We have $m = 2a+1$ for some positive integer a, so $a < m$ is also fantabulous.

■ *Case 2.* The number m is a multiple of 3;
We have $m = 3a$ for some positive integer a, so $a < m$ is also fantabulous.

■ *Case 3.* The number m is 4 modulo 6;
We have $m = 6a - 2$ for some positive integer a. We have the sequence of fantabulous numbers

$$(6a-2) \to (12a-3) \to (4a-1),$$

so $4a - 1 < m$ is also fantabulous.

■ *Case 4.* The number m is 2 modulo 6;

We have $m = 6a + 2$ for some positive integer a. We have the sequence of fantabulous numbers

$$(6a+2) \to (12a+5) \to (36a+15) \to (18a+7) \to (9a+3) \to (3a+1),$$

so $3a + 1 < m$ is also fantabulous.

In all cases, we see that there is another fantabulous number less than m. Since 2021 is fantabulous, it follows that 1 is fantabulous.

Observe that a number m is not fantabulous if and only if all of the elements of the set $\{m, 2m+1, 3m\}$ are not fantabulous. So, the argument above shows that if there exists a positive integer that is not fantabulous, then 1 would not be fantabulous either. This is a contradiction, so all positive integers are fantabulous and, in particular, 2021^{2021} is fantabulous.

Solution 3.

The following transformations show that a is fantabulous if and only if $3a, 3a+1$ or $3a+2$ are fantabulous.

$a \to 3a$

$a \to 2a+1 \to 6a+3 \to 3a+1$

$a \to 2a+1 \to 4a+3 \to 12a+9 \to 36a+27 \to 18a+13 \to 9a+6 \to 3a+2$

This implies that $a \geq 3$ is fantabulous if and only if $f(a) = \left\lfloor \frac{a}{3} \right\rfloor$ is fantabulous. We can use this to deduce that 1 and 2 are fantabulous from the fact that 2021 is fantabulous in the following way:

$$2021 \to 673 \to 224 \to 74 \to 24 \to 8 \to 2 \to 5 \to 1$$

We can apply f sufficiently many times to any positive integer n to arrive at the number 1 or 2. It follows that every positive integer is fantabulous, so 2021^{2021} is fantabulous.

Problem 2. Find all functions $f: \mathbb{Q} \to \mathbb{Q}$ such that the equation

$$f(xf(x) + y) = f(y) + x^2$$

holds for all rational numbers x and y.

Here, $\mathbb{Q}$ denotes the set of rational numbers.

(Slovakia, Patrik Bak)

__Answer__: $f(x) = x$ and $f(x) = -x$.

Solution. Denote the equation from the statement by (1). Let $xf(x) = A$ and $x^2 = B$. The equation (1) is of the form

$$f(A + y) = f(y) + B$$

Also, if we put $y \to -A + y$, we have $f(A - A + y) = f(-A + y) + B$. Therefore

$$f(-A + y) = f(y) - B$$

We can easily show that for any integer n we even have

$$f(nA + y) = f(y) + nB \tag{2}$$

Indeed, it's trivially true for $n = 0$ and if this holds true for some integer n, then

$$f\big((n+1)A + y\big) = f(A + y + nA) = f(ny + A) + B = f(y) + nB + B = f(y) + (n+1)B$$

and

$$f\big((n-1)A + y\big) = f(-A + nA + y) = f(nA + y) - B = f(y) + nB - B = f(y) + (n-1)B.$$

So, equation (2) follows from the induction on n.

Now we can say that for any integer k it holds

$$f(nxf(x) + y) = f(y) + nx^2 \tag{3}$$

If y is given, then $f(y) + nx^2$ can be any rational number, since nx^2 can be any rational number. If it is supposed to be $\frac{p}{q}$, where $q \neq 0$, then we may take $n = pq$, and $x = \frac{1}{q}$. Therefore f is surjective on $\mathbb{Q}$. So there's a rational number c such that $f(c) = 0$. Be putting $x = c$ into (1) we immediately get $c = 0$, i.e. $f(0) = 0$. Therefore, $f(x) = 0$ if and only if $x = 0$.

For any integer n and for any rational x, y it holds

$$f(n^2xf(x) + y) = f(y) + n^2x^2 = f(y) + (nx)^2 = f(nxf(nx) + y) \tag{4}$$

After taking $y = -nxf(nx)$ in (4), the right-hand side becomes 0, therefore

$$n^2xf(x) - nxf(nx) = 0.$$

This simplifies into $nf(x) = f(nx)$ for $x \neq 0$, but it also holds for $x = 0$. Therefore, for any rational number $x = \frac{p}{q}$ we have,

$$f(x) = f\left(\frac{p}{q}\right) = f\left(p \cdot \frac{1}{q}\right) = p \cdot f\left(\frac{1}{p}\right) = p \cdot \frac{f\left(q \cdot \frac{1}{q}\right)}{q} = \frac{p}{q} \cdot f(1) = xf(1)$$

So, we have $f(x) = kx$, for some rational number k. Let's put this answer in (1) and we get $k(xkx + y) = ky + x^2$, thus $k^2 = 1$. Therefore $f(x) = x$ and $f(x) = -x$ are solutions.

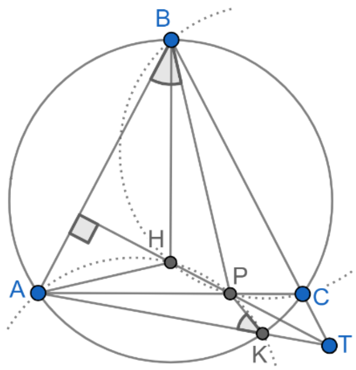
Problem 3. Let ABC be a triangle with an obtuse angle at A. Let E and F be the intersections of the external bisector of angle A with the altitudes of ABC through B and C respectively. Let M and N be the points on the segments EC and FB respectively such that $\angle EMA = \angle BCA$ and $\angle ANF = \angle ABC$. Prove that the points E, F, N, M lie on a circle.

(Ukraine, Anton Trygub)

Solution 1.

The first solution is based on the main Lemma. We present this Lemma with two different proofs.

Lemma: Let ABC be an acute triangle with $AB = BC$. Let P be any point on AC. Line passing through P perpendicular to AB, intersects ray BC in point T. If the line AT intersects the circumscribed circle of the triangle ABC the second time at point K, then $\angle AKP = \angle ABP$.



Proof 1:

Let H be the orthocenter of the triangle ABP. Then

$$\angle BHP = 180^0 - \angle BAC = 180^0 - \angle BCP.$$

So $BHPC$ is cyclic. Then we get

$$TK \cdot TA = TC \cdot TB = TP \cdot TH.$$

So, $AHPK$ is also cyclic. But then

$$\angle AKP = 180^0 - \angle AHP = \angle ABP.$$

Proof 2:

Consider the symmetric points B' and C' of B and C, respectively, with respect to the line PT. It is clear that

$$TC' \cdot TB' = TC \cdot TB = TK \cdot TA.$$

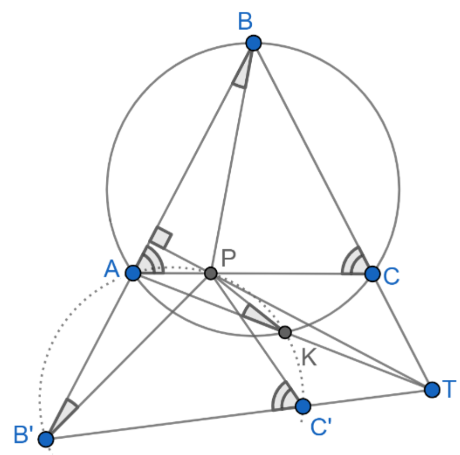
So $B'C'KA$ is cyclic. Also, because of the symmetry we have

$$\angle PC'B' = \angle PCB = \angle PAB.$$

So $B'C'PA$ is also cyclic. Therefore, the points B', C', K, P and A all lie on the common circle. Because of this fact and because of the symmetry again we have

$$\angle PKA = \angle PB'A = \angle PBA.$$

So, lemma is proved and now return to the problem.



Let H be intersection point of the altitudes at B and C. Denote by M' and N' the intersection points of the circumcircle of the triangle HEF with the segments EC and FB, respectively. We are going to show that $M = M'$ and $N = N'$ and it will prove the points E, F, N, M lie on a common circle.

Of course, A is an orthocenter of the triangle BCH. Therefore $\angle BHA = \angle BCA$, $\angle CHA = \angle CBA$ and $\angle HBA = \angle HCA$. Thus

$$\angle HEF = \angle HBA + \angle EAB = \angle HCA + \angle FAC = \angle HFE.$$

So, the triangle HEF is isosceles, $HE = HF$.

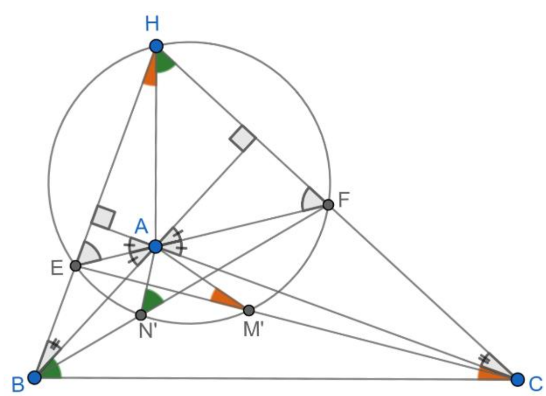
By using lemma, we get

$$\angle AM'E = \angle AHE = \angle ACB,$$

and

$$\angle AN'F = \angle AHF = \angle ABC.$$

Therefore $M = M'$ and $N = N'$ and we are done.



Solution 2.

Let X, Y be projections of B on AC, and C on AB, respectively. Let ω be circumcircle of $BXYC$. Let Z be intersection of EC and ω and D be projection of E on BA.

$$\angle MAC = \angle AME - \angle MCA = \angle XCB - \angle XCE = \angle ZCB = \angle ZXB$$

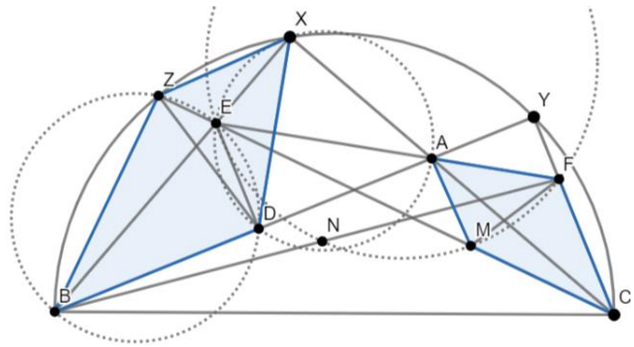
Since $BXYC$ is cyclic $\angle ACY = \angle XBA$, and since $DEXA$ is cyclic

$$\angle EXD = \angle EAD = \angle FAC.$$

Therefore, we get that the quadrangles $BZXD$ and $CMAF$ are similar. Hence $\angle FMC = \angle DZB$. Since $ZEDB$ is cyclic,

$$\angle DZB = \angle DEB = \angle XAB.$$

Thus $\angle FMC = \angle XAB$. Similarly, $\angle ENB = \angle YAC$. We get that $\angle FMC = \angle ENB$ and it implies that the points E, F, N, M lie on a circle.



Problem 4. Let ABC be a triangle with incentre I and let D be an arbitrary point on the side BC. Let the line through D perpendicular to BI intersect CI at E. Let the line through D perpendicular to CI intersect BI at F. Prove that the reflection of A in the line EF lies on the line BC.

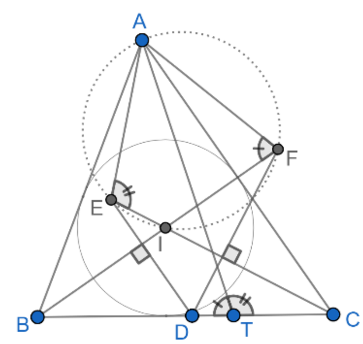
(Australia, Sampson Wong)

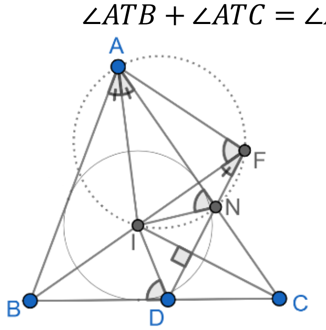
Solution 1.

Let us consider the case when I lies inside of triangle EFD. For the other cases the proof is almost the same only with the slight difference.

We are going to prove that the intersection point of the circumcircles of AEC and AFB (denote it by T) lies on the line BC and this point is the symmetric point of A with respect to EF. First of all we prove that $AEIF$ is cyclic, which implies that T lies on the line BC, because

$$\angle ATB + \angle ATC = \angle AFB + \angle AEC = \angle AFI + \angle AEI = 180^0.$$





Denote by N an intersection point of the lines DF and AC. Of course N is the symmetric point with respect to CI. Thus, $\angle INA = \angle IDB$. Also,

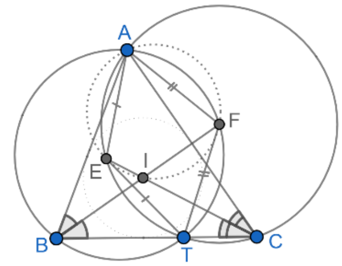
$$\angle IFD = \angle NDC - \angle IBC = 90^0 - \angle ICB - \angle IBC = \angle IAN.$$

So, we get that A, I, N and F lie on a common circle. Therefore, we have $\angle AFI = \angle INA = \angle IDB$. Analogously, $\angle AEI = \angle IDC$ and we have

$$\angle AFI + \angle AEI = \angle IDB + \angle IDC$$

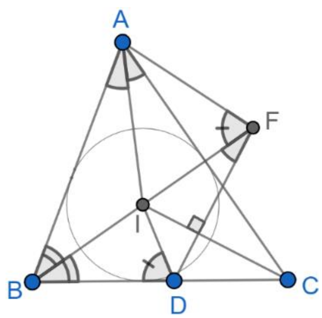
So $\angle AFI + \angle AEI = 180^0$, thus $AEIF$ is cyclic and T lies on the line BC.

Because EC bisects the angle ACB and $AETC$ is cyclic we get $EA = ET$. Because of the similar reasons we have $FA = FT$. Therefore T is the symmetric point of A with respect to the line EF and it lies on the line BC.



Solution 2.

Like to the first solution, consider the case when I lies inside of triangle EFD. we need to prove that $AEIF$ is cyclic. The finish of the proof is the same.



first note that $\triangle FDB \sim \triangle AIB$, because $\angle FBD = \angle ABI$, and

$$\angle BFD = \angle FDC - \angle IBC = 90^0 - \angle ICD - \angle IBC = \angle IAB.$$

Because of the similarity we have $\frac{AB}{AF} = \frac{BI}{BD}$. This equality of the length ratios with $\angle IBD = \angle ABF$ implies that $\triangle ABF \sim \triangle IBD$. Therefore, we have $\angle IDB = \angle AFB$. Analogously, we can get $\angle IDC = \angle AEC$, thus

$$\angle AFI + \angle AEI = \angle IDB + \angle IDC = 180^0.$$

So, $AEIF$ is cyclic and we are done.

Problems 5. A plane has a special point O called the origin. Let P be a set of 2021 points in the plane, such that

(i) no three points in P lie on a line and

(ii) no two points in P lie on a line through the origin.

A triangle with vertices in P is *fat*, if O is strictly inside the triangle. Find the maximum number of *fat* triangles.

(Austria, Veronika Schreitter)

Answer: $2021 \cdot 505 \cdot 337$

Solution

We will count minimal number of triangles that are not fat. Let F set of fat triangles, and S set of triangles that are not fat. If triangle $XYZ \in S$, we call X and Z *good* vertices if OY is located between OX and OZ. For $A \in P$ let $S_A \subseteq S$ be set of triangles in S for which A is one of the good vertex.

It is easy to see that

$$2|S| = \sum_{A \in P} |S_A| \qquad (1)$$

For $A \in P$, let $R_A \subset P$ and $L_A \subset P$ be parts of $P \setminus \{A\}$ divided by AO. Suppose for $AXY \in S$ vertex A is good, then clearly $X, Y \in R_A$ or $X, Y \in L_A$. On the other hand, if $X, Y \in R_A$ or $X, Y \in L_A$ then clearly $AXY \in S$ and A is its good vertex. Therefore,

$$|S_A| = \binom{|R_A|}{2} + \binom{|L_A|}{2} \tag{2}$$

It is easy to show following identity:

$$\frac{x(x-1)}{2} + \frac{y(y-1)}{2} - 2 \cdot \frac{\frac{x+y}{2}\left(\frac{x+y}{2} - 1\right)}{2} = \frac{(x-y)^2}{4} \tag{3}$$

By using (2) and (3) we get

$$|S_A| \geq 2 \cdot \binom{\frac{|R_A| + |L_A|}{2}}{2} = 2 \cdot \binom{1010}{2} = 1010 \cdot 1009 \tag{4}$$

and the equality holds when $|R_A| = |L_A| = 1010$. Hence

$$|S| = \frac{\sum_{A \in P} |S_A|}{2} \geq \frac{2021 \cdot 1010 \cdot 1009}{2} = 2021 \cdot 505 \cdot 1009. \tag{5}$$

Therefore,

$$|F| = \binom{2021}{3} - |S| \leq 2021 \cdot 1010 \cdot 673 - 2021 \cdot 505 \cdot 1009 = 2021 \cdot 505 \cdot 337. \tag{6}$$

For configuration of points on regular 2021-gon which is centered at O, inequalities in (4), (5), (6) become equalities. Hence $2021 \cdot 505 \cdot 337$ is indeed the answer.

Problems 6. Does there exist a nonnegative integer a for which the equation

$$\left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \cdots + \left\lfloor \frac{m}{m} \right\rfloor = n^2 + a$$

has more than one million different solutions (m, n) where m and n are positive integers?
(*The expression* $\lfloor x \rfloor$ *denotes the integer part (or floor) of the real number* x. *Thus* $\lfloor \sqrt{2} \rfloor = 1, \lfloor \pi \rfloor = \left\lfloor \frac{22}{7} \right\rfloor = 3, \lfloor 42 \rfloor = 42$ *and* $\lfloor 0 \rfloor = 0$)

(Austria, Veronika Schreitter)

Answer: Yes.

Solution.

Denote the equation from the statement by (1). The left hand side of (1) depends only on m, and will throughout be denoted by $L(m)$. Fix an integer $q > 10^7$ and note that for $m = q^3$

$$L(q^3) = \sum_{k=1}^{q^3} \left[\frac{q^3}{k}\right] \le \sum_{k=1}^{q^3} \frac{q^3}{k} \le q^3 \cdot \sum_{k=1}^{q^3} \frac{1}{k} \le q^3 \cdot q = q^4. \tag{2}$$

Indeed, the first inequality results from $[x] \le x$. The second inequality can be seen (for instance) as follows. We divide the terms in the sum $\sum_{k=1}^{q^3} \frac{1}{k}$ into several groups: For $j \ge 0$, the j-th group contains the 2^j consecutive terms $\frac{1}{2^j}, \dots, \frac{1}{2^{j+1}-1}$. Since every term in the j-th group is bounded by $\frac{1}{2^j}$, the overall contribution of the j-th group to the sum is at most 1. Since the first q groups together would contain $2^q - 1 > q^3$ terms, the number of groups does not exceed q, and hence the value of the sum under consideration is indeed bounded by q.

Call an integer m *special*, if it satisfies $1 \le L(m) \le q^4$. Denote by $g(m) \ge 1$ the largest integer whose square is bounded by $L(m)$; in other words $g^2(m) \le L(m) < (g(m) + 1)^2$. Note that $g(m) \le q^2$ for all special m, which implies

$$0 \le L(m) - g^2(m) < (g(m) + 1)^2 - g^2(m) = 2g(m) + 1 \le 2q^2 + 1. \tag{3}$$

Finally, we do some counting. Inequality (2) and the monotonicity of $L(m)$ imply that there exist at least q^3 special integers. Because of (3), every special integer m has $0 \le L(m) - g^2(m) \le 2q^2 + 1$. By averaging, at least $\frac{q^3}{2q^2+2} > 10^6$ special integers must yield the same value $L(m) - g^2(m)$. This frequently occurring value is our choice for α, which yields more than 10^6 solutions $(m, g(m))$ to equation (1). Hence, the answer to the problem is YES.