Problem 5. The numbers $p$ and $q$ are prime and satisfy
\[
\frac{p}{p+1} + \frac{q+1}{q} = \frac{2n}{n+2}
\]
for some positive integer $n$. Find all possible values of $q - p$.

Problem 6. There are infinitely many people registered on the social network Mugbook. Some pairs of (different) users are registered as friends, but each person has only finitely many friends. Every user has at least one friend. (Friendship is symmetric; that is, if $A$ is a friend of $B$, then $B$ is a friend of $A$.)

Each person is required to designate one of their friends as their best friend. If $A$ designates $B$ as her best friend, then (unfortunately) it does not follow that $B$ necessarily designates $A$ as her best friend. Someone designated as a best friend is called a 1-best friend. More generally, if $n > 1$ is a positive integer, then a user is an $n$-best friend provided that they have been designated the best friend of someone who is an $(n-1)$-best friend. Someone who is a $k$-best friend for every positive integer $k$ is called popular.

(a) Prove that every popular person is the best friend of a popular person.

(b) Show that if people can have infinitely many friends, then it is possible that a popular person is not the best friend of a popular person.

Problem 7. Let $ABC$ be an acute-angled triangle with circumcircle $\Gamma$ and orthocentre $H$. Let $K$ be a point of $\Gamma$ on the other side of $BC$ from $A$. Let $L$ be the reflection of $K$ in the line $AB$, and let $M$ be the reflection of $K$ in the line $BC$. Let $E$ be the second point of intersection of $\Gamma$ with the circumcircle of triangle $BLM$. Show that the lines $KH$, $EM$ and $BC$ are concurrent. (The orthocentre of a triangle is the point on all three of its altitudes.)

Problem 8. A word is a finite sequence of letters from some alphabet. A word is repetitive if it is a concatenation of at least two identical subwords (for example, $ababa$ and $abcabc$ are repetitive, but $ababa$ and $aabb$ are not). Prove that if a word has the property that swapping any two adjacent identical letters makes the word repetitive, then all its letters are identical. (Note that one may swap two adjacent identical letters, leaving a word unchanged.)

Language: English

Day: 2

Friday, April 13, 2012

Time: 4 hours and 30 minutes

Each problem is worth 7 points