Problem 4 (Poland). Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.

Solution 1. Let QX, PY be tangent to the incircle of ABC, where X, Y lie on the incircle and do not lie on AC, AB. Denote $\angle BAC = \alpha$, $\angle CBA = \beta$, $\angle ACB = \gamma$.

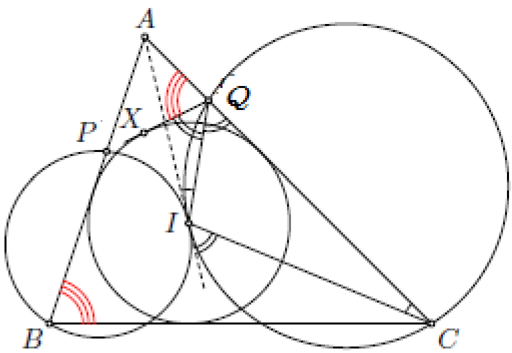
Since AI is tangent to the circumcircle of CQI we get $\angle QIA = \angle QCI = \frac{\gamma}{2}$. Thus

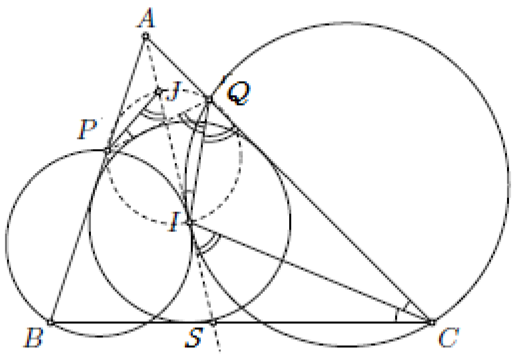
$$\angle IQC = \angle IAQ + \angle QIA = \frac{\alpha}{2} + \frac{\gamma}{2}.$$

By the definition of X we have $\angle IQC = \angle XQI$, therefore

$$\angle AQX = 180^\circ - \angle XQC = 180^\circ - \alpha - \gamma = \beta.$$

Similarly one can prove that $\angle APY = \gamma$. This means that Q, P, X, Y are collinear which leads us to the conclusion that $X = Y$ and QP is tangent to the incircle at X.





Solution 2. By the power of a point we have

$$AD \cdot AC = AI^2 = AP \cdot AB, \quad \text{which means that} \quad \frac{AQ}{AP} = \frac{AB}{AC}$$

and therefore triangles ADP, ABC are similar. Let J be the incenter of AQP. We obtain

$$\angle JPQ = \angle ICB = \angle QCI = \angle QIJ,$$

thus J, P, I, Q are concyclic. Let S be the intersection of AI and BC. It follows that

$$\angle IQP = \angle IJP = \angle SIC = \angle IQC.$$

This means that IQ is the angle bisector of $\angle CQP$, so QP is indeed tangent to the incircle of ABC.

Comment. The final angle chasing from the Solution 2 may simply be replaced by the observation that since J, P, I, Q are concyclic, then I is the A-excenter of triangle APQ.

Solution 3. Like before, notice that $AQ \cdot AC = AP \cdot AB = AI^2$. Consider the positive inversion Ψ with center A and power AI^2. This maps P to B (and vice-versa), Q to C

(and vice-versa), and keeps the incenter I fixed. The problem statement will follow from the fact that the image of the incircle of triangle ABC under Ψ is the so-called mixtilinear incircle of ABC, which is defined to be the circle tangent to the lines AB, AC, and the circumcircle of ABC. Indeed, since the image of the line QP is the circumcircle of ABC, and inversion preserves tangencies, this implies that QP is tangent to the incircle of ABC.

We justify the claim as follows: let γ be the incircle of ABC and let Γ_A be the A-mixtilinear incircle of ABC. Let K and L be the tangency points of γ with the sides AB and AC, and let U and V be the tangency points of Γ_A with the sides AB and AC, respectively. It is well-known that the incenter I is the midpoint of segment UV. In particular, since also $AI \perp UV$, this implies that $AU = AV = \frac{AI}{\cos\frac{A}{2}}$. Note that $AK = AL = AI \cdot \cos\frac{A}{2}$. Therefore, $AU \cdot AK = AV \cdot AL = AI^2$, which means that U and V are the images of K and L under Ψ. Since Γ_A is the unique circle simultaneously tangent to AB at U and to AC at V, it follows that the image of γ under Ψ must be precisely Γ_A, as claimed.

Solution by Achilleas Sinefakopoulos, Greece. From the power of a point theorem, we have

$$AP \cdot AB = AI^2 = AQ \cdot AC.$$

Hence $PBCQ$ is cyclic, and so, $\angle APQ = \angle BCA$. Let K be the circumcenter of $\triangle BIP$ and let L be the circumcenter of $\triangle QIC$. Then $\overline{KL}$ is perpendicular to $\overline{AI}$ at I.

Let N be the point of intersection of line $\overline{KL}$ with $\overline{AB}$.Then in the right triangle $\triangle NIA$, we have $\angle ANI = 90^\circ - \frac{\angle BAC}{2}$ and from the external angle theorem for triangle $\triangle BNI$, we have $\angle ANI = \frac{\angle ABC}{2} + \angle NIB$. Hence

$$\angle NIB = \angle ANI - \frac{\angle ABC}{2} = \left(90^\circ - \frac{\angle BAC}{2}\right) - \frac{\angle ABC}{2} = \frac{\angle BCA}{2}.$$

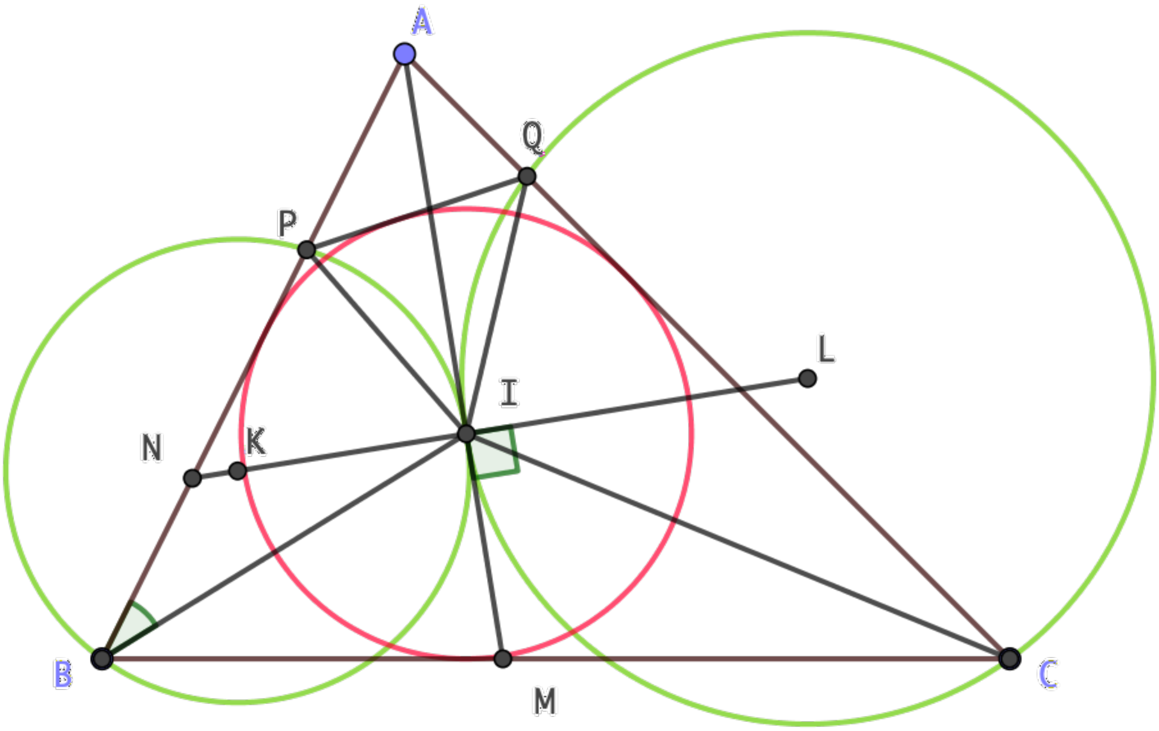
Since MI is tangent to the circumcircle of $\triangle BIP$ at I, we have

$$\angle BPI = \angle BIM = \angle NIM - \angle NIB = 90^\circ - \frac{\angle BCA}{2}.$$

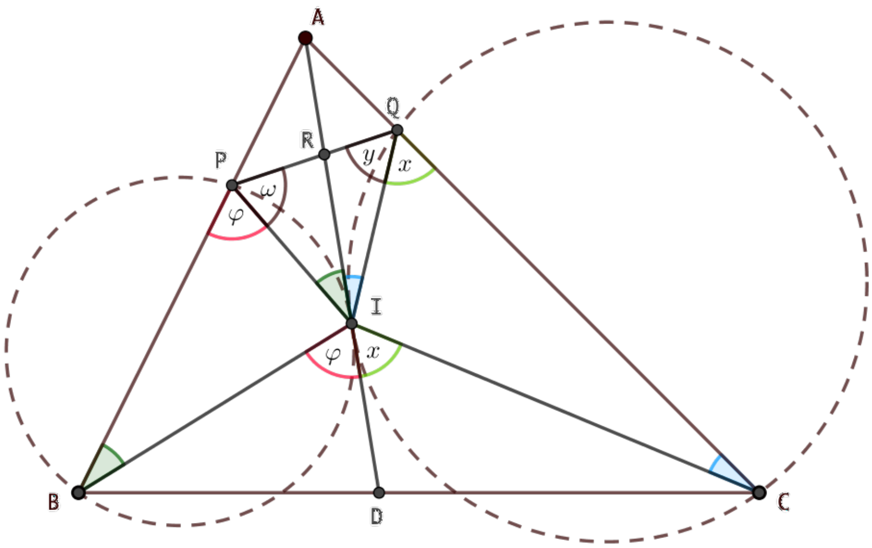
Also, since $\angle APQ = \angle BCA$, we have

$$\angle QPI = 180^\circ - \angle APQ - \angle BPI = 180^\circ - \angle BCA - \left(90^\circ - \frac{\angle BCA}{2}\right) = 90^\circ - \frac{\angle BCA}{2},$$

as well. Hence I lies on the angle bisector of $\angle BPQ$, and so it is equidistant from its sides $\overline{PQ}$ and $\overline{PB}$. Therefore, the distance of I from $\overline{PQ}$ equals the inradius of $\triangle ABC$, as desired.



Solution by Eirini Miliori (HEL2). Let D be the point of intersection of $\overline{AI}$ and $\overline{BC}$ and let R be the point of intersection of $\overline{AI}$ and $\overline{PQ}$. We have $\angle RIP = \angle PBI = \frac{\angle B}{2}$, $\angle RIQ = \angle ICQ = \frac{\angle C}{2}$, $\angle IQC = \angle DIC = x$ and $\angle BPI = \angle BID = \varphi$, since $\overline{AI}$ is tangent to both circles.



From the angle bisector theorem, we have

$$\frac{RQ}{RP} = \frac{AQ}{AP} \quad \text{and} \quad \frac{AC}{AB} = \frac{DC}{BD}.$$

Since $\overline{AI}$ is tangent to both circles at I, we have $AI^2 = AQ \cdot AC$ and $AI^2 = AP \cdot AB$. Therefore,

$$\frac{RQ}{RP} \cdot \frac{DC}{BD} = \frac{AQ \cdot AC}{AB \cdot AP} = 1. \tag{1}$$

From the sine law in triangles $\triangle QRI$ and $\triangle PRI$, it follows that $\frac{RQ}{\sin \frac{\angle C}{2}} = \frac{RI}{\sin y}$ and $\frac{RP}{\sin \frac{\angle B}{2}} = \frac{RI}{\sin \omega}$, respectively. Hence

$$\frac{RQ}{RP} \cdot \frac{\sin \frac{\angle B}{2}}{\sin \frac{\angle C}{2}} = \frac{\sin \omega}{\sin y}. \tag{2}$$

Similarly, from the sine law in triangles $\triangle IDC$ and $\triangle IDB$, it is $\frac{DC}{\sin x} = \frac{ID}{\sin \frac{\angle C}{2}}$ and $\frac{BD}{\sin \varphi} = \frac{ID}{\sin \frac{\angle B}{2}}$, and so

$$\frac{DC}{BD} \cdot \frac{\sin \varphi}{\sin x} = \frac{\sin \frac{\angle B}{2}}{\sin \frac{\angle C}{2}}. \tag{3}$$

By multiplying equations (2) with (3), we obtain $\frac{RQ}{RP} \cdot \frac{DC}{BD} \cdot \frac{\sin \varphi}{\sin x} = \frac{\sin \omega}{\sin y}$, which combined with (1) and cross-multiplying yields

$$\sin \varphi \cdot \sin y = \sin \omega \cdot \sin x. \tag{4}$$

Let $\theta = 90^\circ + \frac{\angle A}{2}$. Since I is the incenter of $\triangle ABC$, we have $x = 90^\circ + \frac{\angle A}{2} - \varphi = \theta - \phi$. Also, in triangle $\triangle PIQ$, we see that $\omega + y + \frac{\angle B}{2} + \frac{\angle C}{2} = 180^\circ$, and so $y = \theta - \omega$.

Therefore, equation (4) yields

$$\sin \varphi \cdot \sin(\theta - \omega) = \sin \omega \cdot \sin(\theta - \varphi),$$

or

$$\frac{1}{2}\left(\cos(\varphi - \theta + \omega) - \cos(\varphi + \theta - \omega)\right) = \frac{1}{2}\left(\cos(\omega - \theta + \varphi) - \cos(\omega + \theta - \varphi)\right),$$

which is equivalent to

$$\cos(\varphi + \theta - \omega) = \cos(\omega + \theta - \varphi).$$

So

$$\varphi + \theta - \omega = 2k \cdot 180^\circ \pm (\omega + \theta - \varphi), \quad (k \in \mathbb{Z}.)$$

If $\varphi + \theta - \omega = 2k \cdot 180^\circ + (\omega + \theta - \varphi)$, then $2(\varphi - \omega) = 2k \cdot 180^\circ$, with $|\varphi - \omega| < 180^\circ$ forcing $k = 0$ and $\varphi = \omega$. If $\varphi + \theta - \omega = 2k \cdot 180^\circ - (\omega + \theta - \varphi)$, then $2\theta = 2k \cdot 180^\circ$, which contradicts the fact that $0^\circ < \theta < 180^\circ$. Hence $\varphi = \omega$, and so PI is the angle bisector of $\angle QPB$.

Therefore the distance of I from $\overline{PQ}$ is the same with the distance of I from AB, which is equal to the inradius of $\triangle ABC$. Consequently, $\overline{PQ}$ is tangent to the incircle of $\triangle ABC$.

Problem 5 (Netherlands).

Let $n \geq 2$ be an integer, and let $a_1, a_2, \dots, a_n$ be positive integers. Show that there exist positive integers $b_1, b_2, \dots, b_n$ satisfying the following three conditions:

1. $a_i \leq b_i$ for $i = 1, 2, \dots, n$;
2. the remainders of $b_1, b_2, \dots, b_n$ on division by n are pairwise different; and
3. $b_1 + \cdots + b_n \leq n \left(\dfrac{n-1}{2} + \left\lfloor \dfrac{a_1 + \cdots + a_n}{n} \right\rfloor \right)$.

(Here, $\lfloor x \rfloor$ denotes the integer part of real number x, that is, the largest integer that does not exceed x.)

Solution 1. We define the b_i recursively by letting b_i be the smallest integer such that $b_i \geq a_i$ and such that b_i is not congruent to any of $b_1, \dots, b_{i-1}$ modulo n. Then $b_i - a_i \leq i - 1$, since of the i consecutive integers $a_i, a_i + 1, \dots, a_i + i - 1$, at most $i - 1$ are congruent to one of $b_1, \dots, b_{i-1}$ modulo n. Since all b_i are distinct modulo n, we have $\sum_{i=1}^n b_i \equiv \sum_{i=1}^n (i-1) = \frac{1}{2}n(n-1)$ modulo n, so n divides $\sum_{i=1}^n b_i - \frac{1}{2}n(n-1)$. Moreover, we have $\sum_{i=1}^n b_i - \sum_{i=1}^n a_i \leq \sum_{i=1}^n (i-1) = \frac{1}{2}n(n-1)$, hence $\sum_{i=1}^n b_i - \frac{1}{2}n(n-1) \leq \sum_{i=1}^n$. As the left hand side is divisible by n, we have

$$\frac{1}{n} \left(\sum_{i=1}^n b_i - \frac{1}{2} n (n-1) \right) \leq \left\lfloor \frac{1}{n} \sum_{i=1}^n a_i \right\rfloor$$

which we can rewrite as

$$\sum_{i=1}^n b_i \leq n \left(\frac{n-1}{2} + \left\lfloor \frac{1}{n} \sum_{i=1}^n a_i \right\rfloor \right)$$

as required.

Solution 2. Note that the problem is invariant under each of the following operations:

- adding a multiple of n to some a_i (and the corresponding b_i);
- adding the same integer to all a_i (and all b_i);
- permuting the index set $1, 2, \dots, n$.

We may therefore remove the restriction that our a_i and b_i be positive.

For each congruence class $\overline{k}$ modulo n ($\overline{k} = \overline{0}, \dots, \overline{n-1}$), let $h(k)$ be the number of i such that a_i belongs to $\overline{k}$. We will now show that the problem is solved if we can find a $t \in \mathbb{Z}$ such that

$$\begin{array}{rcl} h(t) & \geq & 1, \\ h(t) + h(t+1) & \geq & 2, \\ h(t) + h(t+1) + h(t+2) & \geq & 3, \\ & \vdots & \end{array}$$

Indeed, these inequalities guarantee the existence of elements $a_{i_1} \in \overline{t}$, $a_{i_2} \in \overline{t} \cup \overline{t+1}$, $a_{i_3} \in \overline{t} \cup \overline{t+1} \cup \overline{t+2}$, et cetera, where all i_k are different. Subtracting appropriate multiples of n and reordering our elements, we may assume $a_1 = t$, $a_2 \in \{t, t+1\}$, $a_3 \in \{t, t+1, t+2\}$, et cetera. Finally subtracting t from the complete sequence, we may assume $a_1 = 0$, $a_2 \in \{0,1\}$, $a_3 \in \{0,1,2\}$ et cetera. Now simply setting $b_i = i-1$ for all i suffices, since $a_i \le b_i$ for all i, the b_i are all different modulo n, and

$$\sum_{i=1}^{n} b_i = \frac{n(n-1)}{2} \le \frac{n(n-1)}{2} + n\left[\frac{\sum_{i=1}^{n} a_i}{n}\right].$$

Put $x_i = h(i) - 1$ for all $i = 0, \ldots, n-1$. Note that $x_i \ge -1$, because $h(i) \ge 0$. If we have $x_i \ge 0$ for all $i = 0, \ldots, n-1$, then taking $t = 0$ completes the proof. Otherwise, we can pick some index j such that $x_j = -1$. Let $y_i = x_i$ where $i = 0, \ldots, j-1, j+1, \ldots, n-1$ and $y_j = 0$. For sequence $\{y_i\}$ we have

$$\sum_{i=0}^{n-1} y_i = \sum_{i=0}^{n-1} x_i + 1 = \sum_{i=0}^{n-1} h(i) - n + 1 = 1,$$

so from Raney's lemma there exists index k such that $\sum_{i=k}^{k+j} y_i > 0$ for all $j = 0, \ldots, n-1$ where $y_{n+j} = y_j$ for $j = 0, \ldots, k-1$. Taking $t = k$ we will have

$$\sum_{t=k}^{k+i} h(t) - (i+1) = \sum_{t=k}^{k+i} x(t) \ge \sum_{t=k}^{k+i} y(t) - 1 \ge 0,$$

for all $i = 0, \ldots, n-1$ and we are done.

Solution 3. Choose a random permutation $c_1, \ldots, c_n$ of the integers $1, 2, \ldots, n$. Let $b_i = a_i + f(c_i - a_i)$, where $f(x) \in \{0, \ldots, n-1\}$ denotes a remainder of x modulo n. Observe, that for such defined sequence the first two conditions hold. The expected value of $B := b_1 + \ldots + b_n$ is easily seen to be equal to $a_1 + \ldots + a_n + n(n-1)/2$. Indeed, for each i the random number $c_i - a_i$ has uniform distribution modulo n, thus the expected value of $f(c_i - a_i)$ is $(0 + \ldots + (n-1))/n = (n-1)/2$. Therefore we may find such c that $B \le a_1 + \ldots + a_n + n(n-1)/2$. But $B - n(n-1)/2$ is divisible by n and therefore $B \le n[(a_1 + \ldots + a_n)/n] + n(n-1)/2$ as needed.

Solution 4. We will prove the required statement for all sequences of non-negative integers a_i by induction on n.

Case $n = 1$ is obvious, just set $b_1 = a_1$.

Now suppose that the statement is true for some $n \ge 1$; we shall prove it for $n+1$.

First note that, by subtracting a multiple of $n+1$ to each a_i and possibly rearranging indices we can reduce the problem to the case where $0 \le a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} < n+1$.

Now, by the induction hypothesis there exists a sequence $d_1, d_2, \ldots, d_n$ which satisfies the properties required by the statement in relation to the numbers $a_1, \ldots, a_n$. Set $I = \{i | 1 \le i \le n \text{ and } d_i \bmod n \ge a_i\}$ and construct b_i, for $i = 1, \ldots, n+1$, as follows:

$$b_i = \begin{cases} d_i \bmod n, \text{ when } i \in I, \\ n+1+(d_i \bmod n), \text{ when } i \in \{1, \ldots, n\} \setminus I, \\ n, \text{ for } i = n+1. \end{cases}$$

Now, $a_i \le d_i \bmod n \le b_i$ for $i \in I$, while for $i \notin I$ we have $a_i \le n \le b_i$. Thus the sequence $(b_i)_{i=1}^{n+1}$ satisfies the first condition from the problem statement.

By the induction hypothesis, the numbers $d_i \bmod n$ are distinct for $i \in \{1, \ldots, n\}$, so the values $b_i \bmod (n+1)$ are distinct elements of $\{0, \ldots, n-1\}$ for $i \in \{1, \ldots, n\}$. Since $b_{n+1} = n$, the second condition is also satisfied.

Denote $k = |I|$. We have

$$\sum_{i=1}^{n+1} b_i = \sum_{i=1}^{n} b_i + n = \sum_{i=1}^{n} d_i \bmod n + (n-k)(n+1) + n =$$

$$\frac{n(n+1)}{2} + (n-k)(n+1),$$

hence we need to show that

$$\frac{n(n+1)}{2} + (n-k)(n+1) \le \frac{n(n+1)}{2} + (n+1)\left\lceil \frac{\sum_{i=1}^{n+1} a_i}{n+1} \right\rceil;$$

equivalently, that

$$n - k \le \left\lceil \frac{\sum_{i=1}^{n+1} a_i}{n+1} \right\rceil.$$

Next, from the induction hypothesis we have

$$\frac{n(n-1)}{2} + n\left\lceil \frac{\sum_{i=1}^{n} a_i}{n} \right\rceil \ge \sum_{i=1}^{n} d_i = \sum_{i \in I} d_i + \sum_{i \notin I} d_i \ge$$

$$\sum_{i \in I} d_i \bmod n + \sum_{i \notin I} (n + d_i \bmod n) = \frac{n(n-1)}{2} + (n-k)n$$

or

$$n - k \le \left\lceil \frac{\sum_{i=1}^{n} a_i}{n} \right\rceil.$$

Thus, it's enough to show that

$$\frac{\sum_{i=1}^{n} a_i}{n} \le \frac{\sum_{i=1}^{n+1} a_i}{n+1}$$

because then

$$n - k \le \left\lceil \frac{\sum_{i=1}^{n} a_i}{n} \right\rceil \le \left\lceil \frac{\sum_{i=1}^{n+1} a_i}{n+1} \right\rceil.$$

But the required inequality is equivalent to $\sum_{i=1}^{n} a_i \le n a_{n+1}$, which is obvious.

Solution 5. We can assume that all $a_i \in \{0, 1, \ldots, n-1\}$, as we can deduct n from both a_i and b_i for arbitrary i without violating any of the three conditions from the problem statement. We shall also assume that $a_1 \le \ldots \le a_n$.

Now let us provide an algorithm for constructing $b_1, \ldots, b_n$.

We start at step 1 by choosing $f(1)$ to be the maximum i in $\{1,\ldots,n\}$ such that $a_i \leq n-1$, that is $f(1) = n$. We set $b_{f(1)} = n-1$.

Having performed steps 1 through j, at step $j+1$ we set $f(j+1)$ to be the maximum i in $\{1,\ldots,n\} \setminus \{f(1),\ldots,f(j)\}$ such that $a_i \leq n-j-1$, if such an index exists. If it does, we set $b_{f(j+1)} = n-j-1$. If there is no such index, then we define $T = j$ and assign to the terms b_i, where $i \notin f(\{1,\ldots,j\})$, the values $n, n+1\ldots, 2n-j-1$, in any order, thus concluding the run of our algorithm.

Notice that the sequence $(b_i)_{i=1}^n$ satisfies the first and second required conditions by construction. We wish to show that it also satisfies the third.

Notice that, since the values chosen for the b_i's are those from $n-T$ to $2n-T-1$, we have

$$\sum_{i=1}^{n} b_i = \frac{n(n-1)}{2} + (n-T)n.$$

It therefore suffices to show that

$$\left\lceil \frac{a_1 + \ldots + a_n}{n} \right\rceil \geq n - T,$$

or (since the RHS is obviously an integer) $a_1 + \ldots + a_n \geq (n-T)n$.

First, we show that there exists $1 \leq i \leq T$ such that $n - i = b_{f(i)} = a_{f(i)}$.

Indeed, this is true if $a_n = n-1$, so we may suppose $a_n < n-1$ and therefore $a_{n-1} \leq n-2$, so that $T \geq 2$. If $a_{n-1} = n-2$, we are done. If not, then $a_{n-1} < n-2$ and therefore $a_{n-2} \leq n-3$ and $T \geq 3$. Inductively, we actually obtain $T = n$ and necessarily $f(n) = 1$ and $a_1 = b_1 = 0$, which gives the desired result.

Now let t be the largest such index i. We know that $n - t = b_{f(t)} = a_{f(t)}$ and therefore $a_1 \leq \ldots \leq a_{f(t)} \leq n-t$. If we have $a_1 = \ldots = a_{f(t)} = n-t$, then $T = t$ and we have $a_i \geq n-T$ for all i, hence $\sum_i a_i \geq n(n-T)$. Otherwise, $T > t$ and in fact one can show $T = t + f(t+1)$ by proceeding inductively and using the fact that t is the *last* time for which $a_{f(t)} = b_{f(t)}$.

Now we get that, since $a_{f(t+1)+1} \geq n-t$, then $\sum_i a_i \geq (n-t)(n-f(t+1)) = (n-T+f(t+1))(n-f(t+1)) = n(n-T)+nf(t+1)-f(t+1)(n-T+f(t+1)) = n(n-T)+tf(t+1) \geq n(n-T)$.

Greedy algorithm variant 1 (ISR). Consider the residues $0,\ldots,n-1$ modulo n arranged in a circle clockwise, and place each a_i on its corresponding residue; so that on each residue there is a stack of all a_is congruent to it modulo n, and the sum of the sizes of all stacks is exactly n. We iteratively flatten and spread the stacks forward, in such a way that the a_is are placed in the nearest available space on the circle clockwise (skipping over any already flattened residue or still standing stack). We may choose the order in which the stacks are flattened. Since the total amount of numbers equals the total number of spaces, there is always an available space and at the end all spaces are covered. The b_is are then defined by adding to each a_i the number of places it was moved forward, which clearly satifies (i) and (ii), and we must prove that they satisfy (iii) as well.

Suppose that we flatten a stack of k numbers at a residue i, causing it to overtake a stack of l numbers at residue $j \in (i, i+k)$ (we can allow j to be larger than n and identify it with its residue modulo n). Then in fact in fact in whichever order we would flatten the two stacks, the total number of forward steps would be the same, and the total sum of the corresponding b_t (such that $a_t \bmod n \in \{i,j\}$) would be the same. Moreover, we can merge the stacks to a single stack of $k+l$ numbers at residue i, by replacing each $a_t \equiv j \pmod{n}$ by $a'_t = a_t - (j-i)$, and this stack would be flattened forward into the same positions as the separate stacks would have been, so applying our algorithm to the new stacks will yield the same total sum of $\sum b_i$ – but the a_is are strictly decreased, so $\sum a_i$ is decreased, so $\left\lfloor \frac{\sum a_i}{n} \right\rfloor$ is not increased – so by merging the stacks, we can only make the inequality we wish to prove tighter.

Thus, as long as there is some stack that when flattened will overtake another stack, we may merge stacks and only make the inequality tighter. Since the amount of numbers equals the amount of places, the merging process terminates with stacks of sizes $k_1, \dots, k_m$, such that the stack j, when flattened, will exactly cover the interval to the next stack. Clearly the numbers in each such stack were advanced by a total of $\sum_{t=1}^{k_j-1} = \frac{k_j(k_j-1)}{2}$, thus $\sum b_i = \sum a_i + \sum_j \frac{k_j(k_j-1)}{2}$. Writing $\sum a_i = n \cdot r + s$ with $0 \le s < n$, we must therefore show

$$s + \sum_j \frac{k_j(k_j-1)}{2} \le \frac{n(n-1)}{2}.$$

Ending 1. Observing that both sides of the last inequality are congruent modulo n (both are congruent to the sum of all different residues), and that $0 \le s < n$, the inequality is eqivalent to the simpler $\sum_j \frac{k_j(k_j-1)}{2} \le \frac{n(n-1)}{2}$. Since $x(x-1)$ is convex, and k_j are non-negative integers with $\sum_j k_j = n$, the left hand side is maximal when $k_{j'} = n$ and the rest are 0, and then eqaulity is achieved. (Alternatively it follows easily for any non-negative reals from AM-GM.)

Ending 2. If $m = 1$ (and $k_1 = n$), then all numbers are in a single stack and have the same residue, so $s = 0$ and equality is attained. If $m \ge 2$, then by convexity $\sum_j \frac{k_j(k_j-1)}{2}$ is maximal for $m = 2$ and $(k_1, k_2) = (n-1, 1)$, where it equals $\frac{(n-1)(n-2)}{2}$. Since we always have $s \le n-1$, we find

$$s + \sum_j \frac{k_j(k_j-1)}{2} \le (n-1) + \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2}$$

as required.

Greedy algorithm variant 1' (ISR). We apply the same algorithm as in the previous solution. However, this time we note that we may merge stacks not only when they overlap after flattening, but also when they merely touch front-to-back: That is, we relax the condition $j \in (i, i+k)$ to $j \in (i, i+k]$; the argument for why such merges are allowed is exactly the same (But note that this is now sharp, as merging non-touching stacks can cause the sum of b_is to decrease).

We now observe that as long as there at least two stacks left, at least one will spread to touch (or overtake) the next stack, so we can perform merges until there is only one stack left. We are left with verifying that the inequality indeed holds for the case of only one stack which is spread forward, and this is indeed immediate (and in fact equality is achieved).

Greedy algorithm variant 2 (ISR). Let $c_i = a_i \bmod n$. Iteratively define $b_i = a_i + l_i$ greedily, write $d_i = c_i + l_i$, and observe that $l_i \leq n-1$ (since all residues are present in $a_i, \dots, a_i + n - 1$), hence $0 \leq d_i \leq 2n-2$. Let $I = \{i \in I : d_i \geq n\}$, and note that $d_i = b_i \bmod n$ if $i \notin I$ and $d_i = (b_i \bmod n) + n$ if $i \in I$. Then we must show

$$\sum (a_i + l_i) = \sum b_i \leq \frac{n(n-1)}{2} + n \left\lfloor \frac{\sum a_i}{n} \right\rfloor$$
$$\iff \sum (c_i + l_i) \leq \sum (b_i \bmod n) + n \left\lfloor \frac{\sum c_i}{n} \right\rfloor$$
$$\iff n|I| \leq n \left\lfloor \frac{\sum c_i}{n} \right\rfloor \iff |I| \leq \left\lfloor \frac{\sum c_i}{n} \right\rfloor \iff |I| \leq \frac{\sum c_i}{n}$$

Let $k = |I|$, and for each $0 \leq m < n$ let $J_m = \{i : c_i \geq n - m\}$. We claim that there must be some m for which $|J_m| \geq m + k$ (clearly for such m, at least k of the sums d_j with $j \in J_m$ must exceed n, i.e. at least k of the elements of J_m must also be in I, so this m is a "witness" to the fact $|I| \geq k$). Once we find such an m, then we clearly have

$$\sum c_i \geq (n-m)|J_m| \geq (n-m)(k+m) = nk + m(n - (k+m)) \geq nk = n|I|$$

as required. We now construct such an m explicitly.

If $k = 0$, then clearly $m = n$ works (and also the original inequality is trivial). Otherwise, there are some d_is greater than n, and let $r + n = \max d_i$, and suppose $d_t = r + n$ and let $s = c_t$. Note that $r < s < r + n$ since $l_t < n$. Let $m \geq 0$ be the smallest number such that $n - m - 1$ is not in $\{d_1, \dots, d_t\}$, or equivalently m is the largest such that $[n - m, n) \subset \{d_1, \dots, d_t\}$. We claim that this m satisfies the required property. More specifically, we claim that $J'_m = \{i \leq t : d_i \geq n - m\}$ contains exactly $m + k$ elements and is a subset of J_m.

Note that by the greediness of the algorithm, it is impossible that for $[c_i, d_i)$ to contain numbers congruent to $d_j \bmod n$ with $j > i$ (otherwise, the greedy choice would prefer d_j to d_i at stage i). We call this the *greedy property*. In particular, it follows that all i such that $d_i \in [s, d_t) = [c_t, d_t)$ must satisfy $i < t$. Additionally, $\{d_i\}$ is disjoint from $[n + r + 1, 2n)$ (by maximality of d_t), but does intersect every residue class, so it contains $[r + 1, n)$ and in particular also $[s, n)$. By the greedy property the latter can only be attained by d_i with $i < t$, thus $[s, n) \subset \{d_1, \dots, d_t\}$, and in particular $n - m \leq s$ (and in particular $m \geq 1$).

On the other hand $n - m > r$ (since $r \notin \{d_i\}$ at all), so $n - m - 1 \geq r$. It follows that there is a time $t' \geq t$ for which $d_{t'} \equiv n - m - 1 \pmod{n}$: If $n - m - 1 = r$ then this is true for $t' = t$ with $d_t = n + r = 2n - m - 1$; whereas if $n - m - 1 \in [r + 1, n)$ then there is some t' for which $d_{t'} = n - m - 1$, and by the definition of m it satisfies $t' > t$.

Therefore for all $i < t \leq t'$ for which $d_i \geq n - m$, necessarily also $c_i \geq n - m$, since otherwise $d_{t'} \in [c_i, d_i)$, in contradiction to the greedy property. This is also true for $i = t$, since $c_t = s \geq n - m$ as previously shown. Thus, $J'_m \subset J_m$ as claimed.

Finally, since by definition of m and greediness we have $[n - m, n) \cup \{d_i : i \in I\} \subset \{d_1, \dots, d_t\}$, we find that $\{d_j : j \in J'_m\} = [n - m, n) \cup \{d_i : i \in I\}$ and thus $|J'_m| = |[n - m, n)| + |I| = m + k$ as claimed.

Problem 6 (United Kingdom).

On a circle, Alina draws 2019 chords, the endpoints of which are all different. A point is considered *marked* if it is either

(i) one of the 4038 endpoints of a chord; or

(ii) an intersection point of at least two chords.

Alina labels each marked point. Of the 4038 points meeting criterion (i), Alina labels 2019 points with a 0 and the other 2019 points with a 1. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive).

Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with k marked points has $k-1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference.

Alina finds that the $N+1$ yellow labels take each value $0, 1, \ldots, N$ exactly once. Show that at least one blue label is a multiple of 3.

(A *chord* is a line segment joining two different points on a circle.)

Solution 1. First we prove the following:

Lemma: *if we color all of the points white or black, then the number of white-black edges, which we denote E_{WB}, is equal modulo 2 to the number of white (or black) points on the circumference, which we denote C_W, resp. C_B.*

Observe that changing the colour of any interior point does not change the parity of E_{WB}, as each interior point has even degree, so it suffices to show the statement holds when all interior points are black. But then $E_{WB} = C_W$ so certainly the parities are equal.

Now returning to the original problem, assume that no two adjacent vertex labels differ by a multiple of three, and three-colour the vertices according to the residue class of the labels modulo 3. Let E_{01} denote the number of edges between 0-vertices and 1-vertices, and C_0 denote the number of 0-vertices on the boundary, and so on.

Then, consider the two-coloring obtained by combining the 1-vertices and 2-vertices. By applying the lemma, we see that $E_{01} + E_{02} \equiv C_0 \mod 2$.

$$\text{Similarly } E_{01} + E_{12} \equiv C_1, \quad \text{and } E_{02} + E_{12} \equiv C_2, \quad \text{mod } 2.$$

Using the fact that $C_0 = C_1 = 2019$ and $C_2 = 0$, we deduce that either E_{02} and E_{12} are even and E_{01} is odd; or E_{02} and E_{12} are odd and E_{01} is even.

But if the edge labels are the first N non-negative integers, then $E_{01} = E_{12}$ unless $N \equiv 0$ modulo 3, in which case $E_{01} = E_{02}$. So however Alina chooses the vertex labels, it is not possible that the multiset of edge labels is $\{0, \ldots, N\}$.

Hence in fact two vertex labels must differ by a multiple of 3.

Solution 2. As before, colour vertices based on their label modulo 3.

Suppose this gives a valid 3-colouring of the graph with 2019 0s and 2019 1s on the circumference. Identify pairs of 0-labelled vertices and pairs of 1-labelled vertices on the circumference, with one 0 and one 1 left over. The resulting graph has even degrees except these two leaves. So the connected component $\mathcal{C}$ containing these leaves has an Eulerian path, and any other component has an Eulerian cycle.

Let E_{01}^* denote the number of edges between 0-vertices and 1-vertices in $\mathcal{C}$, and let E_{01}' denote the number of such edges in the other components, and so on. By studying whether a given vertex has label congruent to 0 modulo 3 or not as we go along the Eulerian path in $\mathcal{C}$, we find $E_{01}^* + E_{02}^*$ is odd, and similarly $E_{01}^* + E_{12}^*$ is odd. Since neither start nor end vertex is a 2-vertex, $E_{02}^* + E_{12}^*$ must be even.

Applying the same argument for the Eulerian cycle in each other component and adding up, we find that $E_{01}' + E_{02}'$, $E_{01}' + E_{12}'$, $E_{02}' + E_{12}'$ are all even. So, again we find $E_{01} + E_{02}$, $E_{01} + E_{12}$ are odd, and $E_{02} + E_{12}$ is even, and we finish as in the original solution.